THE APRIL MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-fifth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 2–3, 1915, it being the fourth regular meeting of the Society at Chicago. Seventy-five persons were in attendance upon the various sessions, including the following fifty-three members of the Society:

Professor G. A. Bliss, Professor P. P. Boyd, Professor J. W. Bradshaw, Dr. Josephine E. Burns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. W. Chittenden, Dr. G. R. Clements, Professor H. E. Cobb, Mr. L. C. Cox, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Professor W. B. Ford, Professor A. B. Frizell, Professor E. R. Hedrick, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor Kurt Laves, Professor C. E. Love, Dr. W. V. Lovitt, Professor A. C. Lunn, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Dr. T. E. Mason, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor S. W. Reaves, Professor H. L. Rietz, Mr. George Rutledge, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor D. T. Wilson, Professor A. E. Young, Professor J. W. A. Young, Mr. C. H. Yeaton, Professor Alexander Ziwet.

Professor F. R. Moulton, Vice-President of the Society, presided at the sessions on Friday morning and Saturday afternoon, and Professor E. J. Wilczynski at those on Friday afternoon and Saturday morning.

The social gathering and dinner on Friday evening was attended by forty-seven members.

The following papers were presented at this meeting:

(1) Professor C. H. SISAM: "On sextic surfaces having a nodal curve of order nine."

(2) Professor A. F. CARPENTER: "A theorem on the flecnode curves of a ruled surface."

(3) Professor C. T. SULLIVAN: "Analytic characterization of surfaces having a degenerate directrix quadric."

(4) Professor L. E. DICKSON: "Modular quartic curves."

(5) Dr. W. V. LOVITT: "A type of singular points for a transformation on three variables."

(6) Dr. W. V. LOVITT: "Singularities of contact transformations in two-space."

(7) Professor E. J. WILCZYNSKI: "Relations between the theory of congruences and the theory of surfaces."

(8) Address by former chairman of the Chicago Section Professor E. H. MOORE: "Report on integral equations in general analysis."

(9) Dr. G. E. WAHLIN: "A new development of the theory of algebraic domains."

(10) Professor G. A. BLISS: "Jacobi's condition for problems of the calculus of variations."

(11) Professor C. E. LOVE: "On linear difference and differential equations."

(12) Mr. W. L. HART: "Theorems on functions of infinitely many variables" (preliminary report).

(13) Professor F. R. MOULTON: "The solution of an infinite system of differential equations" (preliminary report).

(14) Professor C. N. MOORE: "On the summability of certain types of series."

(15) Professor R. D. CARMICHAEL: "On the solutions of linear homogeneous difference equations."

(16) Professor R. D. CARMICHAEL: "On certain problems in diophantine analysis."

(17) Professor E. B. STOUFFER: "On seminvariants of linear homogeneous differential equations."

(18) Professor W. B. FORD: "On the representation of arbitrary functions by definite integrals."

(19) Professor W. D. MACMILLAN: "The convergence of the power series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{a_{ij}}{i-j\gamma}x_1^i x_2^j$."

(20) Dr. E. W. CHITTENDEN: "On equivalence of K relations."

(21) Dr. E. W. CHITTENDEN: "On the existence of continuous functions."

(22) Professor L. E. DICKSON: "Historical note on the

proof of the quadratic reciprocity law in a posthumous paper by Gauss."

(23) Professor G. A. MILLER: "Independent generators of a group of finite order."

(24) Professor A. B. FRIZELL: "Well ordered sets of infinite permutations."

(25) Mr. C. R. DINES: "On the Hilbert-Schmidt theory in general analysis and the generalization of a theorem by Picard."

(26) Mr. C. R. DINES: "On the instance suggested by the analogy of the sphere and the ellipsoid as regards positive kernels."

(27) Professor E. J. MOULTON: "On the deviation of a rotating compressible fluid mass from a true spheroid."

(28) Professor R. D. CARMICHAEL: "On the use of functional equations in diophantine analysis."

The papers of Professors Sullivan, Stouffer, and Miller, and the first paper by Dr. Lovitt were read by title. Mr. Hart was introduced by Professors Moore and Moulton. Abstracts of the papers follow, the numbers corresponding to those in the list of titles above.

1. In this paper, Professor Sisam points out some fundamental properties of the unruled sextic surface having a nodal curve of order nine, and of the sextic surface in space of six dimensions of which it is the projection.

2. By determining the conditions satisfied by the coordinates of a plane fixed in space, when referred to a moving tetrahedron of reference, Professor Carpenter proves the following theorem: Let an arbitrary fixed plane P cut the generators of a ruled surface S and let π be the pole of P with respect to the quadric osculating S along a generator g. Then the tangent at π to the locus of π cuts g in a point which is harmonically separated from the point in which P cuts g by the flecnodes of g. A special case of this theorem is stated by M. A. Demoulin in the *Comptes Rendus*, June, 1908, page 1381.

3. The integral surfaces of a completely integrable system of partial differential equations

$$\frac{\partial^2 y}{\partial u^2} + 2b\frac{\partial y}{\partial v} + fy = 0, \quad \frac{\partial^2 y}{\partial v^2} + 2a'\frac{\partial y}{\partial u} + gy = 0,$$

where the fundamental invariants a' and b are characterized by the value

$$a' = b = \frac{1}{2} \frac{\sqrt{\left(\frac{dU}{dy}\right)\left(\frac{dV}{dv}\right)}}{(U+V)},$$

constitute the class of surfaces whose asymptotic curves belong to linear complexes. In a former paper (in volume 15 of the *Transactions*), Dr. Sullivan studied these surfaces and showed that they were the envelopes of either of two one-parameter families of ruled surfaces $[R(u), R(v)]$ with straight line directrices; and that the loci of the directrices of $[R(u), R(v)]$ were complementary reguli of a quadric surface (directrix quadric).

The present paper is devoted to the exceptional case where the directrix quadric degenerates into two planes, distinct or coincident. It is shown that the nature of the directrix quadric is completely characterized by the forms of the two invariants θ/a'^2 and θ'/b^2, which played such a fundamental rôle in the determination of the canonical forms of the seminvariants $f(u, v)$ and $g(u, v)$.

It is found that the most general forms of these invariants are:

1. Directrix quadric non-degenerate,

$$\left(\frac{\theta}{a'^2}\right) = \frac{a_1U^2 + b_1U + c}{(dU/du)}, \quad \left(\frac{\theta'}{b^2}\right) = \frac{a_1V^2 - b_1V + c}{(dV/dv)},$$

where a_1, b_1 and c are constants.

2. Directrix quadric degenerates into two distinct planes,

$$a_1 = b_1 = 0, \quad c \neq 0.$$

3. Directrix quadric degenerates into two coincident planes,

$$a_1 = b_1 = c = 0.$$

The plane of case 3 is identical with that determined by the plane net of lines of the degenerate directrix congruence of the first kind discussed in section 7 of the paper cited above. In this case the finite equations of the surfaces have been obtained by Professor Wilczynski ("Flächen mit unbestimmten Direktrixkurven," *Mathematische Annalen*, December, 1914).

4. Professor Dickson's paper will appear in the current volume of the *American Journal of Mathematics*.

5. It is the purpose of Dr. Lovitt's first paper to discuss from a geometrical standpoint, as far as possible, the character of a transformation

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

in the neighborhood of a special type of point at which the Jacobian of the transformation vanishes.

The initial assumptions are such that attention is confined to non-singular points both of the surface defined by the Jacobian of the transformation and of the transform of this surface by means of equations (1). In section two it is shown that under our assumptions there is a one-to-one correspondence between either of two regions in uvw-space on opposite sides of the Jacobian surface J and their transforms in the xyz-space, the transformed regions being on the same side of a surface J_1 which is the transform of the surface J by means of equations (1). Moreover each of the transformed regions is a connected region possessing interior points. The equations of transformation define inverse functions which are continuous at points of the xyz-space which are on the surface J_1. In section three most of the results of the preceding section are extended to a transformation of a neighborhood of a portion of a singular surface not necessarily restricted to a neighborhood of a particular point. In section four most of the results of the preceding section are extended to a transformation in n variables.

6. In his second paper, Dr. Lovitt studies the effect of the transformation

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p)$$

upon the contact of curves

$$(1) \qquad x = f(t), \quad y = g(t), \quad y' = g'/f' = h(t)$$

in the neighborhood of a set of values $(x, y, p = x_0, y_0, p_0)$ for which the functional determinant of the transformation vanishes, but the matrix of the determinant is of rank two. Let the transformed curves be given by

$$x_1 = f_1(t), \quad y_1 = g_1(t), \quad y_1' = g_1'/f_1' = h_1(t),$$

where f_1, g_1, h_1 are defined by means of the equations of transformation and equations (1). Let us designate as critical

curves that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \cdots, k-1,$$

while $[f_1^{(j)}]^2 + [h_1^{(j)}]^2 \neq 0$.

Then the necessary and sufficient condition that two of the critical curves shall be transformed into curves with second order contact is that the critical curves have contact of order $k+1$.

7. In a paper presented to the Society at Chicago in December, 1914, Professor Wilczynski introduced a number of new concepts into the theory of congruences, among which the axis and ray congruences, and the axis and ray curves were especially important. On the other hand he has introduced into the theory of surfaces the notion of the directrix congruences and the directrix curves. In the present paper the theory of congruences is enriched by formulas and theorems expressing the relations which exist between the directrix curves and directrix congruences of the focal surface, and the other properties of the congruence.

9. In volume 144 of *Crelle's Journal* Hensel published a paper entitled "Ueber die Grundlagen einer neuen Theorie der quadratischen Zahlkörper." Dr. Wahlin's paper is a generalization of this development to any algebraic domain of degree n.

In the first place the ring $R(q, \alpha)$, consisting of the g-adic algebraic numbers determined by the root α of an equation of degree n, is considered. It is shown how the study of these numbers can be reduced to the case where g is a rational prime p and we have the ring $R(p, \alpha)$.

A method is then devised by means of which the further study of this ring can be reduced to the study of certain domains constructed out of the numbers of $R(p, \alpha)$ with a different condition for equality.

10. There are two well-known methods of deducing Jacobi's necessary condition in the calculus of variations. One is a geometric proof not easily extensible to higher spaces, which excludes important exceptional cases; the other depends upon complicated manipulations of the second variation. For the problem in parametric form in the plane the reduction

of the second variation was devised by Weierstrass and is a remarkable piece of analysis. It is, however, very artificial, and also unnecessary if one has in view only the proof of Jacobi's condition. For problems in parametric form in higher spaces a discussion of the second variation has been made by von Escherich by methods in part quite unsymmetrical. The lack of symmetry is due to the division of an arc

$$y_i = y_i(t) \quad (t_1 \leqq t \leqq t_2;\ i = 1, 2, \cdots, n)$$

into a finite number of pieces on each of which at least one of the derivatives $y_i'(t)$ is different from zero, a device which leads to inelegant complications, though his results are symmetric in form. In the paper of Professor Bliss a proof of Jacobi's condition is given which applies with equal simplicity to the parametric case in the plane or in higher spaces. It makes use of the second variation without reduction of any sort, is symmetric in all the variables, and includes the exceptional cases which probably would with difficulty be covered by an extension to higher spaces of the geometric proof mentioned above.

11. The first part of Professor Love's paper is an extension of certain theorems of Ford* on linear difference equations. The behavior, for large positive integral values of x, of n linearly independent solutions of the equation

$$Y(x) = [a_0 + \alpha_0(x)]y(x+n) + [a_1 + \alpha_1(x)]y(x+n-1) + \cdots + [a_n + \alpha_n(x)]y(x) = 0$$

is determined, provided the functions $\alpha_0(x)$, $\alpha_1(x)$, $\cdots$, $\alpha_n(x)$ vanish at infinity to a sufficiently high order. It should be noted that no restriction whatever is laid upon the roots of the characteristic equation

$$a_0\mu^n + a_1\mu^{n-1} + \cdots + a_n = 0,$$

except that a_0 and a_n are assumed different from zero.

The second part of the paper contains the parallel investigation for linear differential equations. The same problem under narrower conditions has been treated by Dini.†

* *Annali di Matematica*, ser. 3, vol. 13 (1907), pp. 263–328.

† *Annali di Matematica*, ser. 3, vol. 2 (1899), pp. 297–324; *ibid.*, vol. 3 (1900), pp. 125–183.

In each part of the paper the results obtained are also applied to the study of the non-homogeneous equation

$$Y(x) = X(x),$$

under suitable restrictions on the function $X(x)$.

12. By a function of infinitely many real variables will be understood a function $f(\xi)$ where ξ represents $(x_1, x_2, \cdots)$. In much of the current work, the function f is defined for values ξ for which $\Sigma_{i=1}^{i=\infty} |x_i^2|$ converges. In Mr. Hart's paper the functions considered are supposed to be defined for values ξ satisfying

$$M_1^{(i)} \leqq x_i \leqq M_2^{(i)} \quad (M_1^{(i)}, M_2^{(i)} \leqq M; i = 1, 2, \cdots).$$

The function f is said to be completely continuous at the point ξ_0 if, whenever

$$\lim_{n=\infty} x_{i,n} = x_{i,0} \qquad (i = 1, 2, \cdots),$$

then

$$\lim_{n=\infty} f(\xi_n) = f(\xi_0) \quad (\xi_n = x_{1,n}, x_{2,n}, \cdots).$$

In addition to certain general theorems on such continuous functions, among which is included an extension of the mean value theorem, there is considered the solution, for ξ in terms of u, of the infinite system of equations

$$f_i(\eta) = 0 \quad (\eta = u, x_1, x_2, \cdots; i = 1, 2, \cdots), \tag{1}$$

where the values of η lie in the region R defined by

$$|\eta - \eta_0| \leqq d_0.$$

The fundamental theorem of implicit function theory is obtained for (1) by a method of successive approximations.

13. Professor Moulton treats an infinite system of differential equations of the analytic type, and, under suitable hypotheses, establishes the existence of an analytic solution. The solution is also studied as a function of the parameters which may be involved and of the initial conditions. The paper is incomplete in that the details of the extension of the solution to the boundary of the region where the initial equations are defined and regular have not been fully worked out.

14. Chapman has obtained sufficient conditions that a series summable (Ck) for any $k > 0$ should be reduced to convergence when certain convergence factors are introduced into the terms of the series. Somewhat earlier G. H. Hardy discussed the conditions under which a series summable (Ck), where k is a positive integer, should remain summable (Ck) when the terms are multiplied by factors of a certain type. In Professor Moore's paper sufficient conditions are obtained that a series summable (Ck) for any $k > 0$ should remain summable (Ck) when certain factors are introduced into the terms.

15. In his first paper Professor Carmichael gives a new method of deriving the general existence theorems for the case of the linear homogeneous difference equation

$$f(x+n) + a_1(x)f(x+n-1) + \cdots + a_n(x)f(x) = 0$$

in which the coefficients $a_1(x), \cdots, a_n(x)$ have the character of rational functions at infinity. By means of the descending formal power series solutions which this equation is known to possess in general a separation of it into two members is effected so that the method of successive approximation is available for deriving a single solution. By repeated use of a method of transformation the remaining solutions of two fundamental sets are then obtained. The analytic character of the functions in one of these sets and their asymptotic character at infinity are determined.

Of the previously developed methods for dealing with the problem of this paper that of the author's dissertation is most closely related to the present one, but the two methods are essentially distinct.

16. By means of a class of algebraic relations due to Fermat and Lagrange and developed in part by Legendre, Professor Carmichael, in his second paper, suggests a general method by which large classes of problems in diophantine analysis become amenable to systematic treatment. The method is applied to the resolution of a considerable number of diophantine equations of interest. The essential element of the method may be described thus: The problem of solving a diophantine equation consists essentially in determining numbers of a certain class (those defined by the form of one member) so that they shall have an additional property (that determined

by the form of the other member). The method suggested and employed in the present paper is suitably to extend the set of numbers defined by the form in one member (sometimes by those in both) so that the extended set shall have the property that the product of any two numbers in the set shall also be in the set. This gives rise to a generalized problem. The latter is solved and the result is then specialized so as to afford the solution of the original problem. This method, though very simple in its characteristics, leads to a considerable number of results of interest.

17. Consider the system of linear homogeneous differential equations

$$y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0, \quad (i = 1, 2, \cdots n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. The most general transformations leaving such a system unchanged in form are given by

$$(1) \qquad y_i = \sum_{k=1}^{n} \alpha_{ik}\bar{y}_k, \qquad |\alpha_{ik}| \neq 0, \quad (i = 1, 2, \cdots n),$$

$$(2) \qquad x = \xi(x),$$

where α_{ik} and ξ are arbitrary functions of x. A function of p_{ik} and q_{ik} and their derivatives which has the same value for the given system as for any system obtained from it by the transformation (1) is called a seminvariant. Professor Stouffer obtains for each value of n a single seminvariant from which the complete system for that value of n may be obtained by successive applications of certain operators. These operators are similar to the Aronhold operator for algebraic invariants. This method of obtaining the seminvariants avoids the solution of complicated systems of partial differential equations.

18. Let $f(x)$ be an arbitrary function of the real variable x defined throughout the interval $(-\pi, \pi)$. Having assigned the special value α $(-\pi < \alpha < \pi)$ to x, let the Fourier series for $f(\alpha)$ be formed. The sum of its first $n+1$ terms may be put in the form

$$(1) \qquad S(n, \alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\psi(n, x - \alpha)dx,$$

where

$$\psi(n, x-\alpha) = \frac{\sin \frac{1}{2}(2n+1)(x-\alpha)}{2 \sin \frac{1}{2}(x-\alpha)},$$

and it is a familiar fact that we then have, at least in general,

$$\lim_{n=\infty} S(n, \alpha) = \frac{f(\alpha-0)+f(\alpha+0)}{2}. \tag{2}$$

The following general problem is thus suggested: Given an arbitrary function $f(x)$ defined throughout any finite interval (a, b) and consider the integral

$$I(n, \alpha) = \int_a^b f(x)\varphi(n, x-\alpha)dx, \tag{3}$$

where $\varphi(n, x-\alpha)$ is now regarded as an unknown function to be so determined that we shall have, analogously to (2),

$$\lim_{n=\infty} I(n, \alpha) = \frac{f(\alpha-0)+f(\alpha+0)}{2} \quad (a<\alpha<b). \tag{4}$$

What conditions upon $\varphi(n, x-\alpha)$ will suffice to insure (4)? This problem was first considered and answered to some extent by Du Bois-Reymond, later it was treated in great detail by Dini in his "Serie di Fourier," and of late years it has been the object of a paper by Hobson.* The conditions in question have thus been in large measure determined, but it does not appear that explicit forms for $\varphi(x, n-\alpha)$ that satisfy such conditions have been worked out to any appreciable extent. In this connection, Professor Ford's paper shows how an infinity of such functions $\varphi(n, x-\alpha)$ may be explicitly obtained.

19. In this paper by Professor MacMillan it is shown that if the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x_1^i x_2^j$$

is convergent for $|x_1| < x_{10}$, $|x_2| < x_{20}$, then the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_{ij}}{i-j\gamma} x_1^i x_2^j$$

also is convergent if $|x_1| < x_{10}$, $|x_2| < x_{20}$, and γ is a positive irrational number which satisfies certain mild conditions.

* *Proc. London Math. Soc.*, vol. 6 (1908), pp. 349–374.

Two applications of these series are given. The first application is to the function

$$W = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^i \mu^j}{i - j\gamma},$$

for which the positive real axis is a line of essential singularities. It is shown that this function is continuous across this axis at such irrational points. The second application is to the existence of a solution of the linear partial differential equation

$$x_1 \frac{\partial \varphi}{\partial x_1} - \gamma x_2 \frac{\partial \varphi}{\partial x_2} = p_1 \varphi + p_2,$$

where p_1 and p_2 are convergent ordinary power series in x_1 and x_2. It is shown that solutions exist as ordinary power series in x_1 and x_2.

20. For a class $\mathfrak{P}$ relations Kp_1p_2m, $\overline{K}p_1p_2m$ are supposed to be defined. These relations are of the type K_2 of Moore's Introduction to General Analysis. Dr. Chittenden defines equivalence for the relations K, $\overline{K}$ and shows that for any unsymmetric K possessing the properties 167 as defined by T. H. Hildebrandt* a symmetric equivalent relation $\overline{K}$ may be defined. The theory of classes $\mathfrak{P}$ and functions on classes $\mathfrak{P}$ as developed by Hildebrandt for an unsymmetric K^{167} relation is equivalent to the corresponding theory for symmetric K relations.

21. Hans Hahn has shown that there exists a non-constant continuous function on every class $\mathfrak{P}$ which admits a definition of voisinage.† Dr. Chittenden shows by modifying the methods of Hahn that this result may be extended to classes which admit a definition of distance subject to the condition $L_n(p_n, q_n) = 0$, $L_n(p_n, p) = 0$ imply $L_n(q_n, p) = 0$. It follows, as shown by Hahn, that, if limit is unique, a necessary and sufficient condition that every continuous function on $\mathfrak{P}$ be bounded and attain its bounds is that $\mathfrak{P}$ be extremal. This result is extended to the case of non-unique limits.

* *American Journal of Mathematics*, vol. 34 (1912), pp. 243–4.
† *Monatshefte für Mathematik und Physik*, vol. 19 (1908), p. 251–5.

22. The seventh proof by Gauss of the quadratic reciprocity theorem occurs in his posthumous paper, Werke 2, 1863, pages 233–5 (Gauss-Maser, pages 623–4). Gauss left only a meager outline of a possible proof, not revised by him for publication. In the version by Bachmann, Niedere Zahlentheorie, 1, 1902, pages 396–9, resort is had (page 399) a second time to the cyclotomic theory, whereas one needs only the fact that in all cases $S(i)$ and $S_1(i)$ are the roots of the congruence at the bottom of page 398; moreover, Bachmann introduces Galois imaginaries, whereas their use was carefully avoided by Gauss (Disquisitiones Arithmeticae, article 338; Werke, 2, page 217). The object of the note by Professor Dickson is to complete the proof in accord with the spirit of the MS. of Gauss. In *Crelle*, 19, 1839, pages 299–306, Schonemann gave a sketch (neither very clear nor full) of a similar proof based on the reality of the roots of the quadratic congruence corresponding to the cyclotomic equation for two periods. Another proof similar to that by Gauss and published three years prior to his is that by V. A. Lebesgue, *Comptes Rendus*, 51, 1860, pages 9–13; it is based on Kummer's theorem on the reality of the roots of the congruence corresponding to a cyclotomic equation for e periods.

23. The term independent generator has been used mainly with reference to abelian groups. Hence it has been commonly used with the restricted meaning that the group generated by any number of a set of these generators has only identity in common with the group generated by the rest of those in the same set. With this restricted meaning Frobenius and Stickelberger proved in 1879 that the number of the independent generators is an invariant of any abelian group whose order is the power of a prime number. It has been proved recently that the number of these independent generators is an invariant of *every* group whose order is a power of a prime when the independent generators are defined as follows: A set of operators $s_1, s_2, \cdots, s_\lambda$ belonging to the group G is called a set of independent generators of G provided that these λ operators generate G but no $\lambda - 1$ of them generate G.

Among the theorems established by Professor Miller in the present paper are the following: A necessary and sufficient condition that a group G contains at least one set of independent generators composed of as many operators as there are

prime factors in the order of the group, is that each of the Sylow subgroups of G is abelian and is generated by a set of operators of prime order each of which is transformed only into powers of itself by every Sylow subgroup whose order is a power of a smaller prime number. The number of the groups of order p^m, p being any prime number, which possess a set of $m-1$ independent generators is either $\frac{1}{2}(m-1)$ or $\frac{1}{2}(m-2)+1$ according as m is odd or even.

24. By a process used in a paper read before the Society at New York, February, 1915, Professor Frizell proves that all well ordered types of order are comprehended in those furnished by the permutations of a simply infinite set. That is, given a well ordered set w, there exists, among the permutations of an ω-series, a set ordinally similar to w.

25. Professor Moore has shown that the general Hilbert-Schmidt theory of the integral equation for the complex-valued hermitian kernel may be secured on the foundation Σ_5; viz.,

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}^{LCDD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHPP_0 \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}}),$$

where $\mathfrak{A}$ is the class of all real or of all complex numbers, $\mathfrak{P}$ is a general class of general elements forming a general range, $\mathfrak{M}$ is a class of single-valued functions on the range $\mathfrak{P}$ to $\mathfrak{A}$ with the properties $LCDD_0R$, $\mathfrak{K}$ is defined as the *-composite of $\mathfrak{M}$ with itself, and J is a general functional operation having properties as indicated. The definite property P_0 of the functional operation is the property which, in the classical instance of integration states that if, for a continuous function ξ, we have $\int_0^1 \xi(s)\bar{\xi}(s)ds = 0$, then $\xi(s)$ is identically zero. Mr. Dines shows that the theory as to the existence of characteristic functions and numbers of a hermitian kernel κ may be obtained on a foundation in which the functional operation is not restricted by postulation of the property P_0. This permits us then to include as an instance that of integration in which the class $\mathfrak{M}$ is taken as the class of all real-valued, bounded functions, integrable with their squares in the sense of Lebesgue. The relation, κ not identically zero, in the classical instance is replaced by the relation $J_{23\ 41}{}^2\kappa\kappa \neq 0$, and the theory is in general the same. The expansion theorem of Hilbert is

stated as giving the expansion of $J^2\alpha\kappa\beta$ where α and β are not necessarily in the class $\mathfrak{M}$ but are functions of a class $\mathfrak{N}$ which contains $\mathfrak{M}$ as a subclass, J being operative on $(\mathfrak{N}\mathfrak{N})_L$ to $\mathfrak{A}$. For example, the class $\mathfrak{N}$ may be taken as the class of all functions, whether bounded or not, integrable with their squares in the sense of Lebesgue.

The unsymmetric kernel is then discussed in a generalization in which not only is the functional operation generalized but also the kernel function is no longer on a composite range necessarily formed by one range with itself but on a composite range formed from two conceptually distinct ranges.

Using these results, Mr. Dines secures a generalization of a theorem due to Picard as to conditions for solvability of an integral equation of the first kind and gives an application to the geometry of a function space.

26. As an instance of the basis Σ_5, Professor Moore has cited that suggested by the analogy of the sphere and the ellipsoid. In this instance, given a function ω, a new functional operation $J\omega$ is obtained from that of Σ_5 by defining the functional operation, operating on a function κ, as $J_{13\,42}{}^2\kappa\omega$. Mercer has shown the importance of the positive kernel as to expansion in a uniformly convergent series of characteristic functions. Mr. Dines discusses the positive kernel in the instance cited above. Conditions on a hermitian function ω are obtained sufficient to secure the equivalence for every hermitian function κ of the properties, positive as to J and positive as to J_ω. There follows a discussion of conditions equivalent to these. The latter part of the work is suggested by that of Fischer on the equivalence of the notions, general and closed, as applied to a unitary and orthogonal set of functions from the class of all functions integrable with their squares in the sense of Lebesgue. The foundation for the discussion is Σ_7, defined as follows:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0RC_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0RB_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \left(\begin{smallmatrix}(\mathfrak{N}\mathfrak{N})L\\ \mathfrak{K}\end{smallmatrix}\right) \text{ to } \mathfrak{A}}).$$

This differs from the foundation Σ_5 in that the functional operation is not restricted by postulation of the property P_0 and that the class $\mathfrak{N}$ is introduced. The property C_J for the class $\mathfrak{N}$ is a closure property, the class being closed as to convergence in the mean.

27. If a homogeneous fluid mass is rotating about an axis with a sufficiently small angular velocity there are two possible oblate spheroid figures of equilibrium, one nearly spherical, Σ_1, and one much flattened at the poles, Σ_2. If the fluid mass is compressible there are two figures of equilibrium nearly oblate spheroids. One approximates Σ_1 and is depressed in middle latitudes below the spheroid having the same polar and equatorial radii. The other approximates Σ_2 and is elevated in middle latitudes. The former theorem was proved by Airy, Callandreau and Darwin; the latter is a new theorem. This is established and a formula for the deviation from a true spheroid is obtained by Professor Moulton. He starts from an assumed relation between the pressure and density in the fluid and by making a compressibility parameter play a fundamental rôle is able to discuss the more flattened figure.

28. In his third paper Professor Carmichael shows how rational solutions of certain functional equations may be employed in solving problems of a certain class in the theory of diophantine analysis. In particular, several problems of Diophantus and Fermat are readily treated. The contents of both this paper and the preceding one by the same author will appear in his forthcoming "Introduction to Diophantine Analysis," to be published by Wiley and Sons.

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Secretary of the Chicago Section.

A GEOMETRIC DERIVATION OF A GENERAL FORMULA FOR THE SOUTHERLY DEVIATION OF FREELY FALLING BODIES.*

BY PROFESSOR WM. H. ROEVER.

(Read before the American Mathematical Society, October 25, 1913.)

WITHIN the last dozen years interest in the problem of the deviations of freely falling bodies seems to have been revived. There is a substantial agreement, among the writers who have treated this subject, as to the magnitude of the easterly deviation, their result being practically the same as that obtained

* See BULLETIN, vol. 20, No. 4, p. 175.

by Gauss more than one hundred years ago. Concerning the southerly deviation, however, there appears to be a diversity of results. This lack of agreement can, for the most part, be accounted for by the different definitions for southerly deviation and the different potential functions for the earth's field of force which have been used in the derivations of the various formulas. In another paper* the author shows that most of these formulas for the southerly deviation are special cases (corresponding to special potential functions and to special definitions) of his general formula

$$\text{(I)} \qquad \text{S.D.} = \left\{ 2\omega^2 \sin 2\phi_0 - 5\left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0},$$

which may be written in the form

$$\text{(II)} \qquad \text{S.D.} = \xi_1 - \xi_2,$$

where

$$\xi_1 = \left\{ 2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0 \right\} \cdot \frac{h^2}{6g_0}, \quad \xi_2 = \left(\frac{\partial g}{\partial \xi}\right)_0 \cdot \frac{h^2}{g_0},$$

S.D. representing the quantity which some writers use for southerly deviation and ξ_1 that which others use. Of the other quantities used in formula (I), h denotes the height through which the body falls, ω the angular velocity of the earth's rotation, ϕ_0, g_0, $(\partial g/\partial \xi)_0$ the values at the point P_0 from which the body falls of the astronomical latitude ϕ, the acceleration g due to weight, and the derivative of g with respect to ξ, where ξ represents distance measured to the south at P_0.

Formula (I), besides including as special cases many formulas which are based on particular potential functions, has a practical advantage over any of these formulas. This the following facts will show. In a country which is not very level the experimentally determined values of the derivative $(\partial g/\partial \xi)$ exceed, in magnitude, many times (sometimes ten or more) the values of this derivative which correspond to generally accepted potential functions. Furthermore, $(\partial g/\partial \xi)$ varies in an apparently very capricious manner. It may be positive at one station and negative at a station a few miles distant.† Therefore, formula (I), with the experimentally

* See the author's paper "Recent work on the deviations of falling bodies," soon to be published in *U. S. Monthly Weather Review*, vol. 43.

† See *Transactions Amer. Math. Society*, vol. 13, pp. 469–490 (Introduction); *Astronomical Journal*, nos. 670–672, Part II, § 4.

determined local value of the derivative $(\partial g/\partial \xi)$, is capable of accounting more accurately for the effect, on the southerly deviation of a falling body, of a local irregularity in the earth's field of force (such as a mountain) than any formula which is based on a particular (and consequently inadequate) potential function.

The author has already shown that, under the assumption of an asymmetric distribution of the earth's gravitating matter as well as under the assumption of a distribution of revolution, formula (I) is the first and predominating term of a still longer expression for the southerly deviation of freely falling bodies.* In view of the generality and practical nature of formula (I) it seems worth while to derive it by still another method. In this article a purely geometric derivation is given under the assumption of a distribution of revolution.

§ 1. *Definitions.*

Let us assume a set of rectangular axes $O - u, v, z$ (Fig. 1)

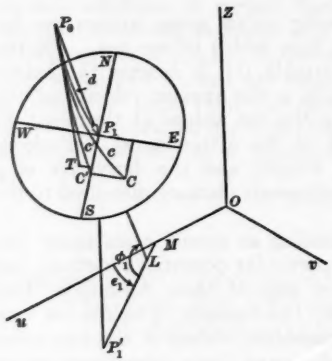

FIG. 1.

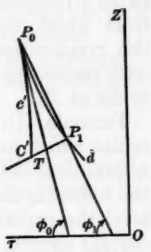

FIG. 2.

which are at rest with respect to the solid part of the earth and such that Oz coincides with the earth's axis of rotation, the positive sense of Oz being from the celestial south pole to the celestial north pole. The field of force which determines

* See *Transactions Amer. Math. Society*, vol. 12, pp. 335–353; vol. 13, pp. 469–490; and *Astronomical Journal*, nos. 670–672, Part II.

weight, i. e., the field in which the plumb-line is in equilibrium, is at rest with respect to these axes. We shall call this field of force the *weight field*. Let us denote by P_1 a point (on or near the earth's surface) which is at rest with respect to the axes $O - u, v, z$. The straight line which passes through P_1 and gives the direction of the force of the weight field at P_1 is defined as the *vertical of* P_1. The vertical of P_1 coincides with the string of a plumb-line, the bob of which is situated at P_1.* The *astronomical meridian plane of* P_1 is the plane which passes through the vertical of P_1 and is parallel to the axis of rotation Oz. The *astronomical latitude of* P_1 is the complement ϕ_1 (Fig. 1) of the angle which the vertical of P_1 (to the zenith) makes with the axis Oz (to the celestial north pole). The *astronomical longitude of* P_1 is the angle ϵ_1 (Fig. 1) which the meridian plane of P_1 makes with a fixed plane (zOu say) through Oz, and is measured from 0° to 360° to the east. The *horizontal plane of* P_1 is the plane which passes through P_1 and is perpendicular to the vertical of P_1. The *north-and-south line of* P_1 is the line of intersection of the meridian and horizontal planes of P_1. The *east-and-west line of* P_1 is the straight line which passes through P_1 and is perpendicular to the meridian plane of P_1. This line may also be defined as the intersection of the horizontal plane of P_1 by the plane which passes through P_1 and is perpendicular to the axis Oz.†

Now let us take a point P_0 (Fig. 1) above P_1 and in the vertical P_1P_0 of P_1. The vertical of P_0 will *not* coincide with the vertical of P_1, but it will pierce the horizontal plane $NESW$ of P_1 in a point T which does not coincide with P_1. Let us suppose a material particle to fall, under the influence of the earth's gravitation, from an initial position of rest (with respect to $O - u, v, z$) at P_0. The path of this particle, with respect to the axes $O - u, v, z$, is a curve c which pierces the horizontal plane $NESW$ of P_1 in a point C.

DEFINITION 1. *The perpendicular distance of the point C from the north-and-south line NP_1S of P_1 is called the easterly deviation of the falling particle, and is regarded as positive to the east* (see Fig. 1).

DEFINITION 2. *The perpendicular distance of the point C*

* It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the bob is a heavy particle.

† The above definitions are practically the same as those given by Pizzetti, Trattato di Geodesia teoretica, § 5 (1905).

from the east-and-west line EP_1W of P_1 is called the southerly deviation of the falling particle, and is regarded as positive to the south (see Fig. 1).

The following definitions will also be found convenient. *A line of force of the weight field* is a curve at each point P of which the tangent is the vertical. *A level (or equipotential) surface of the weight field* is a surface at each point P of which the normal is the vertical. The *plumb-bob locus of the point* P_0 is the locus of all plumb-lines (of the ideal type defined above) which are supported at P_0. Geometrically this locus may be defined in either of the two following ways:

(1) the locus of the points of tangency of tangents drawn from the fixed point P_0 to the lines of force of the weight field.

(2) the locus of the feet of perpendiculars dropped from the fixed point P_0 to the level surfaces of the weight field.

§ 2. *Some Fundamental Relations.*

Let us denote by ω the angular velocity (with respect to the inertial axes of the solar system) of the earth's rotation, and by V the potential function of the earth due to gravitational attraction alone, i. e., the function defined by the integral

$$V = \kappa \int \frac{dm}{\sqrt{(u-a)^2 + (v-b)^2 + (z-c)^2}},$$

where dm represents that element of mass which is situated at the point whose coördinates, with respect to the axes $O - u, v, z$, are a, b, c, the integration being extended throughout the whole volume of the earth, and κ is the gravitation constant. Then the potential function of the weight field is*

$$W = V + \frac{\omega^2}{2}(u^2 + v^2). \tag{1}$$

Then the equations of the level surfaces of the weight field may be written in the form

$$W = K,$$

where K has constant values. The differential equations of the lines of force of the weight field are

$$\frac{du}{W_u} = \frac{dv}{W_v} = \frac{dz}{W_z}, \tag{2}$$

* See Pizzetti, loc. cit., § 2.

where W_u, W_v, W_z stand for the partial derivatives $\partial W/\partial u$, $\partial W/\partial v$, $\partial W/\partial z$ respectively. The equations of the plumb-bob locus of P_0 are

$$\frac{u-u_0}{W_u}=\frac{v-v_0}{W_v}=\frac{z-z_0}{W_z}. \tag{3}$$

From these equations it follows that P_0 (u_0, v_0, z_0) lies on this locus, and also that P_1 (u_1, v_1, z_1) lies on it, for since the straight line P_1P_0 is the vertical of P_1,

$$\frac{u_1-u_0}{W_u^{(1)}}=\frac{v_1-v_0}{W_v^{(1)}}=\frac{z_1-z_0}{W_z^{(1)}},$$

where $W_u^{(1)}$, $W_v^{(1)}$, $W_z^{(1)}$ denote the particular values of W_u, W_v, W_z respectively when $u=u_1$, $v=v_1$, $z=z_1$. It also follows from the systems of equations (2) and (3) that the plumb-bob locus of P_0 (represented by d in Fig. 1) has for tangent at P_0 the vertical P_0T of P_0.* Hence the

* In order to prove this let us write equations (3) in the form

$$F=\begin{vmatrix} v-v_0 & z-z_0 \\ W_v & W_z \end{vmatrix}=0, \quad \Phi=\begin{vmatrix} z-z_0 & u-u_0 \\ W_z & W_u \end{vmatrix}=0.$$

Then for points of this curve

$$du:dv:dz=\begin{vmatrix} F_v & F_z \\ \Phi_v & \Phi_z \end{vmatrix}:\begin{vmatrix} F_z & F_u \\ \Phi_z & \Phi_u \end{vmatrix}:\begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix}.$$

But

$$F_u=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vu} & W_{zu} \end{vmatrix}, \quad F_v=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vv} & W_{zv} \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ W_v & W_z \end{vmatrix},$$

$$F_z=\begin{vmatrix} v-v_0 & z-z_0 \\ W_{vz} & W_{zz} \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ W_v & W_z \end{vmatrix},$$

$$\Phi_u=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zu} & W_{uu} \end{vmatrix}+\begin{vmatrix} 0 & 1 \\ W_z & W_u \end{vmatrix}, \quad \Phi_v=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zv} & W_{uv} \end{vmatrix},$$

$$\Phi_z=\begin{vmatrix} z-z_0 & u-u_0 \\ W_{zz} & W_{uz} \end{vmatrix}+\begin{vmatrix} 1 & 0 \\ W_z & W_u \end{vmatrix}.$$

For the particular point P_0, (u_0, v_0, z_0) these derivatives have the values

$$F_u^{(0)}=0, \quad F_v^{(0)}=W_z^{(0)}, \quad F_z^{(0)}=-W_v^{(0)},$$

$$\Phi_u^{(0)}=-W_z^{(0)}, \quad \Phi_v^{(0)}=0, \quad \Phi_z^{(0)}=W_u^0.$$

Hence at P_0

$$du:dv:dz=W_z^{(0)}W_u^{(0)}:W_z^0W_v^0:W_z^0W_z^0=W_u^0:W_v^0:W_z^0.$$

It is assumed that the derivatives W_u, W_v, W_z do not all vanish at the same point (u, v, z) and that each of them has continuous first partial derivatives.

THEOREM 1. *The plumb-bob locus d of P_0 passes through the points P_0 and P_1 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

If we denote time by t, the differential equations of motion (with respect to the axes $O - u, v, z$) of the falling particle are*

$$\frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} - \omega^2 u = \frac{\partial V}{\partial u},$$

$$\frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} - \omega^2 v = \frac{\partial V}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}. \tag{4}$$

Since by relation (1)

$$\omega^2 u + \frac{\partial V}{\partial u} = \frac{\partial W}{\partial u}, \quad \omega^2 v + \frac{\partial V}{\partial v} = \frac{\partial W}{\partial v}, \quad \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z},$$

equations (4) may be written in the form:†

$$\frac{d^2u}{dt^2} - 2\omega\frac{dv}{dt} = \frac{\partial W}{\partial u}, \quad \frac{d^2v}{dt^2} + 2\omega\frac{du}{dt} = \frac{\partial W}{\partial v}, \quad \frac{d^2z}{dt^2} = \frac{\partial W}{\partial z}. \tag{5}$$

The path c of the falling particle is that solution of equations (5) [or (4)] which is subject to the initial condition

$$\text{when } t = 0, \quad u = u_0, \quad v = v_0, \quad z = z_0,$$

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dz}{dt} = 0. \tag{6}$$

Therefore it follows from equations (5) that

$$\left(\frac{d^2u}{dt^2}\right)_0 = W_u^{(0)}, \quad \left(\frac{d^2v}{dt^2}\right)_0 = W_v^{(0)}, \quad \left(\frac{d^2z}{dt^2}\right)_0 = W_z^{(0)}, \tag{7}$$

where the subscript $_0$ indicates that those particular values of the derivatives d^2u/dt^2, d^2v/dt^2, d^2z/dt^2 have been taken for which $t = 0$. Since $W_u^{(0)}$, $W_v^{(0)}$, $W_z^{(0)}$ are proportional to the directional cosines of the vertical P_0T of P_0, we have, on account of relations (6) and (7), the

* For a derivation of these equations see § 2 of the author's paper in the *Transactions Amer. Math. Society*, vol. 13, pp. 469–490, where x, y, z, U stand for u, v, z, V here used.

† A superficial reading of equations (5) might lead one *erroneously* to the conclusion that ω^2 was neglected. In these equations ω^2 is contained in the expressions of the derivatives $\partial W/\partial u$, $\partial W/\partial v$. In short, equations (5) are the equivalents of equations (4), and *not* the approximations thereto made by neglecting ω^2.

THEOREM 2. *The path c of the falling particle (with respect to the axes $O - u, v, z$) passes through the point P_0 and is tangent at P_0 to the vertical P_0T of P_0* (see Fig. 1).

§ 3. *Special Relations Corresponding to a Distribution of Revolution.*

By a *distribution of revolution* of the earth's gravitating matter we shall mean a distribution of matter for which the potential function (1) assumes the form

$$W = f(\tau, z), \tag{8}$$

where

$$\tau = \sqrt{u^2 + v^2}$$

represents the distance of the point (u, v, z) from the earth's axis of rotation Oz.

For such a distribution the level surfaces of the weight field are surfaces of revolution, the two-parameter family of lines of force of the weight field lie in the one-parameter family of planes which pass through the axis Oz, and the differential equation of those lines of force which lie in any such plane is

$$\frac{dz}{d\tau} = \frac{\partial f/\partial z}{\partial f/\partial \tau}; \tag{9}$$

the plumb-bob locus of P_0 lies in the plane of P_0 and Oz, and its equation in this plane is

$$\frac{\partial f}{\partial z}(\tau - \tau_0) - \frac{\partial f}{\partial \tau}(z - z_0) = 0. \tag{10}$$

The meridian planes of P_1 and P_0 then coincide and contain, besides the verticals of P_1 and P_0, the plumb-bob locus d of P_0. The path c of the falling particle does not lie in this plane. Let us project it orthographically on this plane and denote by c' and C' the projections of c and C respectively (see Fig. 2). Then it follows from Definition 2, § 1, that

$$\text{S.D.} = P_1C', \tag{11}$$

where S.D. stands for southerly deviation. The point T in which the vertical of P_0 pierces the horizontal plane of P_1 now (i. e., for a distribution of revolution) lies in the north-and-south line of P_1. Let us put (see Fig. 2)

$$\xi_1 = TC', \quad \xi_2 = TP_1,$$

the positive sense of each of these quantities being the same as that of S.D. Then

$$(12)\quad \xi_1 - \xi_2 = TC' - TP_1 = P_1T + TC' = P_1C' = \text{S.D.}$$

The curve c' is closely related to a curve c'' which we will now define. The path c of the falling particle may be regarded as the directrix of a surface of revolution of axis Oz. This surface of revolution passes through the point P_0. The intersection of this surface of revolution by the plane of P_0 and Oz is the curve c''.

We will now prove the two following theorems.

THEOREM 3. *The curvature, at P_0, of the plumb-bob locus d of P_0 is twice that of the line of force of the weight field which passes through P_0, and the concavity of these curves is in the same direction.*

THEOREM 4. *The curvature, at P_0, of the curves c' and c'' is the same in both magnitude and direction.*

In order to prove Theorem 3 let us note that, since the two curves are tangent, it will be sufficient to compare the values, which correspond to the two curves, of the second derivative $d^2z/d\tau^2$. For the line of force the expression for this second derivative is found by differentiating equation (9). Thus we find for the line of force that

$$(13)\quad \frac{d^2z}{d\tau^2} = \frac{\dfrac{\partial^2 f}{\partial\tau\partial z}\left\{\left(\dfrac{\partial f}{\partial\tau}\right)^2 - \left(\dfrac{\partial f}{\partial z}\right)^2\right\} + \dfrac{\partial f}{\partial\tau}\dfrac{\partial f}{\partial z}\left\{\dfrac{\partial^2 f}{\partial z^2} - \dfrac{\partial^2 f}{\partial\tau^2}\right\}}{\left(\dfrac{\partial f}{\partial\tau}\right)^3}.$$

Let us represent by $F(\tau, z)$ the left member of equation (10). The derivatives $dz/d\tau$, $d^2z/d\tau^2$ of the function defined by equation (10) are expressible in terms of the partial derivatives of F by means of the well known relations

$$(14)\quad \begin{gathered} \frac{\partial F}{\partial\tau} + \frac{\partial F}{\partial z}\cdot\frac{dz}{d\tau} = 0, \\ \frac{\partial^2 F}{\partial\tau^2} + 2\frac{\partial^2 F}{\partial\tau\partial z}\cdot\frac{dz}{d\tau} + \frac{\partial^2 F}{\partial z^2}\left(\frac{dz}{d\tau}\right)^2 + \frac{\partial F}{\partial z}\cdot\frac{d^2z}{d\tau^2} = 0. \end{gathered}$$

For the particular point $P_0(\tau_0, z_0)$ we have*

* Since $F = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \dfrac{\partial f}{\partial\tau} & \dfrac{\partial f}{\partial z} \end{vmatrix}$,

$$\left(\frac{\partial F}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0, \quad \left(\frac{\partial F}{\partial z}\right)_0 = -\left(\frac{\partial f}{\partial \tau}\right)_0,$$

$$(15)\quad \left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 = 2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \quad \left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2}\right)_0 = -2\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0.$$

Hence for the plumb-bob locus we have, at P_0,

$$\left(\frac{dz}{d\tau}\right)_0 = -\frac{\left(\frac{\partial F}{\partial \tau}\right)_0}{\left(\frac{\partial F}{\partial z}\right)_0} = \frac{\left(\frac{\partial f}{\partial z}\right)_0}{\left(\frac{\partial f}{\partial \tau}\right)_0},$$

$$(16)\quad \left(\frac{d^2 z}{d\tau^2}\right)_0 = \frac{\left(\frac{\partial^2 F}{\partial \tau^2}\right)_0 + 2\left(\frac{\partial^2 F}{\partial \tau \partial z}\right)_0\left(\frac{dz}{d\tau}\right)_0 + \left(\frac{\partial^2 F}{\partial z^2}\right)_0\left(\frac{dz}{d\tau}\right)_0^2}{-\left(\frac{\partial F}{\partial z}\right)_0}$$

$$= 2\frac{\left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0\left\{\left(\frac{\partial f}{\partial \tau}\right)_0^2 - \left(\frac{\partial f}{\partial z}\right)_0^2\right\} + \left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0\left\{\left(\frac{\partial^2 f}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0\right\}}{\left(\frac{\partial f}{\partial \tau}\right)_0^3}.$$

$$\frac{\partial F}{\partial \tau} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial F}{\partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial f}{\partial \tau} & \frac{\partial f}{\partial z} \end{vmatrix},$$

$$\frac{\partial^2 F}{\partial \tau^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau^3} & \frac{\partial^3 f}{\partial z \partial \tau^2} \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z \partial \tau},$$

$$\frac{\partial^2 F}{\partial \tau \partial z} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial^2 \tau \partial z} & \frac{\partial^3 f}{\partial z^2 \partial \tau} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau^2} & \frac{\partial^2 f}{\partial z \partial \tau} \end{vmatrix} + \frac{\partial^2 f}{\partial z^2},$$

$$\frac{\partial^2 F}{\partial z^2} = \begin{vmatrix} \tau - \tau_0 & z - z_0 \\ \frac{\partial^3 f}{\partial \tau \partial z^2} & \frac{\partial^3 f}{\partial z^3} \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ \frac{\partial^2 f}{\partial \tau \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} - \frac{\partial^2 f}{\partial \tau \partial z}.$$

The first of these relations being the same as equation (9) for the point P_0 furnishes another proof of the fact, contained in Theorem 1, that the plumb-bob locus of P_0 is tangent, at P_0, to the line of force (of the weight field) which passes through P_0. This fact, taken in conjunction with the fact that the second of expressions (16) is + 2 times as great as expression (13) for the point P_0, constitutes the proof of Theorem 3.

In order to prove Theorem 4 let us express the equations of motion of the falling particle in terms of cylindrical coordinates. For this purpose let us put

$$u = \tau \cos \lambda, \quad v = \tau \sin \lambda, \quad z = z, \tag{17}$$

where $\tau = \sqrt{u^2 + v^2}$ has the same meaning as above, and λ is the angle which the plane through Oz and the general point (u, v, z) makes with the plane zOu. If we subject the system of equations (4) to this transformation, we obtain the equations

$$\frac{d^2\tau}{dt^2} - \tau\left(\omega + \frac{d\lambda}{dt}\right)^2 = \frac{\partial V}{\partial \tau}, \quad \frac{d}{dt}\left\{\tau^2\left(\omega + \frac{d\lambda}{dt}\right)\right\} = \frac{\partial V}{\partial \lambda}, \tag{18}$$

$$\frac{d^2z}{dt^2} = \frac{\partial V}{\partial z}.$$

By the same transformation relation (1) becomes

$$W = V + \frac{\omega^2}{2}\tau^2. \tag{19}$$

If we assume the potential function to be of the form (8),

$$\frac{\partial W}{\partial \lambda} = \frac{\partial V}{\partial \lambda} = 0, \tag{20}$$

and then the second of equations (18) yields the integral

$$\frac{d\lambda}{dt} + \omega = \frac{k}{\tau^2}, \tag{21}$$

where k is the constant of integration. The first of equations (18) then becomes

$$\frac{d^2\tau}{dt^2} = \frac{\partial V}{\partial \tau} + \frac{k^2}{\tau^3} = \frac{\partial f}{\partial \tau} - \omega^2\tau + \frac{k^2}{\tau^3}. \tag{22}$$

Furthermore, when $t = 0$ we have $d\lambda/dt = 0$ and $\tau = \tau_0$, whence by (21)

$$k = \omega\tau_0^2. \tag{23}$$

If now we put

$$\Omega(\tau, z) = f(\tau, z) - \frac{\omega^2}{2}\left(\tau^2 + \frac{\tau_0^4}{\tau^2}\right), \tag{24}$$

the equations of motion (for a distribution of revolution) take the form

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z}, \quad \frac{d\lambda}{dt} = \omega\left(\frac{\tau_0^2}{\tau^2} - 1\right). \tag{25}$$

The path c of the falling particle is (for a distribution of revolution) that solution of equations (25) which is subject to the initial conditions

$$\begin{gathered}\text{when } t = 0, \quad \tau = \tau_0 = \sqrt{u_0^2 + v_0^2}, \qquad z = z_0, \\ \lambda = \lambda_0 = \arctan\frac{v_0}{u_0}, \quad \frac{dz}{dt} = \frac{d\tau}{dt} = \frac{d\lambda}{dt} = 0.\end{gathered} \tag{26}$$

The curve c'' is now easily seen to be that solution of the pair of differential equations

$$\frac{d^2\tau}{dt^2} = \frac{\partial\Omega}{\partial\tau}, \quad \frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z} \tag{27}$$

which is subject to the initial conditions

$$\text{when } t = 0, \quad \tau = \tau_0, \quad z = z_0, \quad \frac{d\tau}{dt} = \frac{dz}{dt} = 0. \tag{28}$$

For the same ordinate z the abscissas of the two curves c'' and c' differ by the amount

$$\begin{aligned}&\tau - \tau\cos(\lambda - \lambda_0) \\ &\quad = 2\tau\sin^2\tfrac{1}{2}(\lambda - \lambda_0) = \tau[\tfrac{1}{2}(\lambda - \lambda_0)^2 + \cdots],\end{aligned} \tag{29}$$

where z and τ are those solutions of equations (27) which are subject to the conditions (28), or where z, τ, λ are those solutions of equations (25) which are subject to the conditions (26).

The solutions τ, z to terms of order not higher than the second in t are easily seen to be

$$\tau = \tau_0 + \alpha_2 t^2 + \cdots, \quad z = z_0 + \beta_2 t^2 + \cdots, \tag{30}$$

where

$$\alpha_2 = \frac{1}{2}\left(\frac{\partial \Omega}{\partial \tau}\right)_0, \quad \beta_2 = \frac{1}{2}\left(\frac{\partial \Omega}{\partial z}\right)_0,$$

the subscript $_0$ indicating that the values of the derivatives to which it is attached correspond to $\tau = \tau_0$ and $z = z_0$. Hence

$$\frac{\tau_0^2}{\tau^2} = 1 - \frac{2\alpha_2}{\tau_0} t^2 + \cdots,$$

and therefore, by the last of equations (25),

$$\frac{d\lambda}{dt} = \omega\left(-\frac{2\alpha_2}{\tau_0} t^2 + \cdots\right), \tag{31}$$

whence, by (26),

$$\lambda - \lambda_0 = -\tfrac{2}{3}\omega \frac{\alpha_2}{\tau_0} t^3 + \cdots. \tag{32}$$

Therefore the difference (29) becomes

$$\begin{aligned} \tau - \tau \cos(\lambda - \lambda_0) &= (\tau_0 + \alpha_2 t^2 + \cdots)\left(\frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0^2} t^6 + \cdots\right) \\ &= \frac{2}{9}\frac{\alpha_2^2\omega^2}{\tau_0} t^6 + \cdots \qquad (33) \\ &= \frac{2}{9}\frac{\omega^2\alpha_2^2}{\tau_0\beta_2^3}(z - z_0)^3 + \cdots. \end{aligned}$$

Since this difference is of the third order in $(z - z_0)$ it follows that the curves c' and c'' have contact of the second order at the point P_0 and hence we have proved Theorem 4.

§ 4. *Some Known Theorems.*

The following three theorems will be found useful:

THEOREM 5. *In a two-dimensional field of force, for which a potential function exists, the curvature of a line of force is the derivative of the logarithm of the force taken in that direction, on an equipotential curve, in which the force increases.*

THEOREM 6. *If, in a positional field of force, a particle starts from rest, the initial curvature of the path described is one-third of the curvature of the line of force through the initial position.**

* See Kasner's Princeton Colloquium Lectures, p. 9, footnote 2.

THEOREM 7. *If at a point P of a curve (with a continuously turning tangent) at which the curvature is finite (i. e., neither zero nor infinite) the tangent and the inner normal be taken as a set of rectangular axes, then the equation of the curve referred to these axes may be written in the form*

$$y = \frac{1}{2}\cdot\frac{1}{\rho}x^2 + \text{an expression of order three, or higher, in } x,$$

*where x is measured from P along the tangent, y is measured from P along the inner normal and ρ is the radius of curvature at P.**

References to proofs of Theorems 6 and 7 are given in the following footnotes. In order to prove Theorem 5 let us refer the field of force to a set of rectangular axes (x, y) and denote by X, Y the components of the force j of the field. Then the differential equation of the lines of force is

$$\frac{dy}{dx} = \frac{Y}{X}, \tag{34}$$

whence

$$\frac{d^2y}{dx^2} = \frac{X^2\frac{\partial Y}{\partial x} - Y^2\frac{\partial X}{\partial y} + XY\left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right)}{X^3}. \tag{35}$$

On the other hand, the magnitude of the force is

$$j = \sqrt{X^2 + Y^2} = X\cos\phi + Y\sin\phi, \tag{36}$$

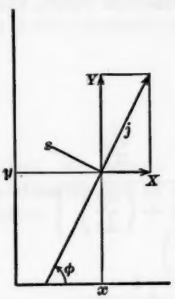

FIG. 3.

FIG. 4.

where $\tan\phi$ is the slope of the vector representing the force

* See Osgood's Introduction to Infinite Series, p. 39. Also Goursat-Hedrick, Mathematical Analysis, vol. 1, p. 451.

(Fig. 3). Hence

$$0 = -X\sin\phi + Y\cos\phi,$$

whence

$$j\cos\phi = X, \quad j\sin\phi = Y.$$

Then

$$\frac{\partial j}{\partial x} = \frac{1}{j}\left(X\frac{\partial X}{\partial x} + Y\frac{\partial Y}{\partial x}\right), \quad \frac{\partial j}{\partial y} = \frac{1}{j}\left(X\frac{\partial X}{\partial y} + Y\frac{\partial Y}{\partial y}\right).$$

If now we represent by s (Figs. 3 and 4) distance measured on a line perpendicular to the vector representing j and such that s is positive on the side opposite to that on which the angle ϕ lies, then

$$\frac{\partial j}{\partial s} = -\frac{\partial j}{\partial x}\sin\phi + \frac{\partial j}{\partial y}\cos\phi = \frac{1}{j}\left(-\frac{\partial j}{\partial x}Y + \frac{\partial j}{\partial y}X\right)$$

(37)
$$= \frac{X^2\frac{\partial X}{\partial y} - Y^2\frac{\partial Y}{\partial x} + XY\left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right)}{j^2}.$$

If now

(38)
$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

which is the case when a potential function exists, then by (35) and (37),

(39)
$$\frac{d^2y}{dx^2} = \frac{j^2}{X^3}\cdot\frac{\partial j}{\partial s},$$

whence

(40)
$$\frac{\partial}{\partial s}\log_e j = \frac{\frac{\partial j}{\partial s}}{j} = \cos^3\phi\,\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}},$$

since

$$\cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

Hence Theorem 5.

§ 5. *Applications of the Preceding Theorems to the Problem under Consideration.*

Let us now replace the variables x and y by τ and z respectively, and denote by U the potential function of the field of force of Theorem 5. Then

$$(41) \qquad j\cos\phi = X = -\frac{\partial U}{\partial \tau}, \quad j\sin\phi = Y = -\frac{\partial U}{\partial z},$$

$$j = \sqrt{\left(\frac{\partial U}{\partial \tau}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2},$$

and relation (37) becomes

$$(42) \qquad \frac{\frac{\partial j}{\partial s}}{j} = \frac{\Theta(U)}{j^3},$$

where

$$\Theta(U) = \frac{\partial^2 U}{\partial \tau \partial z}\left[\left(\frac{\partial U}{\partial z}\right)^2 - \left(\frac{\partial U}{\partial \tau}\right)^2\right] + \frac{\partial U}{\partial \tau}\cdot\frac{\partial U}{\partial z}\left[\frac{\partial^2 U}{\partial \tau^2} - \frac{\partial^2 U}{\partial z^2}\right].$$

If now $U = f(\tau, z)$ is the function (8),

$$(43) \qquad j = g = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2},$$

where g is the acceleration due to weight, formula (42) becomes

$$(44) \qquad \frac{\frac{\partial g}{\partial s}}{g} = \frac{\Theta(f)}{g^3}.$$

If, further, we denote by ξ distance measured along the north-and-south line of P_0 (positive to the south), we obtain the relation

$$(45) \qquad -\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} = \frac{[\Theta(f)]_0}{g_0^3},$$

in which, by Theorem 5 or relation (40), $-(\partial g/\partial \xi)_0/g_0$ is the curvature at P_0 of the line of force of the weight field which passes through P_0. Hence by Theorem 3 we have

THEOREM 8. *The curvature at P_0 of the plumb-bob locus d of P_0 is*

$$-2\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}. \tag{46}$$

Let us next put $U = \Omega$, where Ω is the function defined in (24). By (24)

$$\frac{\partial \Omega}{\partial \tau} = \frac{\partial f}{\partial \tau} - \omega^2\left(\tau - \frac{\tau_0^4}{\tau^3}\right), \quad \frac{\partial \Omega}{\partial z} = \frac{\partial f}{\partial z},$$

$$\frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\partial^2 f}{\partial \tau^2} - \omega^2\left(1 + 3\frac{\tau_0^4}{\tau^4}\right), \quad \frac{\partial^2 \Omega}{\partial \tau \partial z} = \frac{\partial^2 f}{\partial \tau \partial z}, \quad \frac{\partial^2 \Omega}{\partial z^2} = \frac{\partial^2 f}{\partial z^2}, \tag{47}$$

where f is the function (8). For the particular point P_0, whose coordinates are τ_0 and z_0, these derivatives assume the values

$$\left(\frac{\partial \Omega}{\partial \tau}\right)_0 = \left(\frac{\partial f}{\partial \tau}\right)_0, \quad \left(\frac{\partial \Omega}{\partial z}\right)_0 = \left(\frac{\partial f}{\partial z}\right)_0,$$

$$\left(\frac{\partial^2 \Omega}{\partial \tau^2}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau^2}\right)_0 - 4\omega^2, \quad \left(\frac{\partial^2 \Omega}{\partial \tau \partial z}\right)_0 = \left(\frac{\partial^2 f}{\partial \tau \partial z}\right)_0, \tag{48}$$

$$\left(\frac{\partial^2 \Omega}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f}{\partial z^2}\right)_0,$$

whence

$$j_0 = \sqrt{\left(\frac{\partial f}{\partial \tau}\right)_0^2 + \left(\frac{\partial f}{\partial z}\right)_0^2} = g_0,$$

$$\begin{aligned}[\Theta(\Omega)]_0 &= [\Theta(f)]_0 - 4\omega^2\left(\frac{\partial f}{\partial \tau}\right)_0\left(\frac{\partial f}{\partial z}\right)_0 \\ &= [\Theta(f)]_0 - 4\omega^2 g_0^2 \sin\phi_0 \cos\phi_0.\end{aligned} \tag{49}$$

Therefore for the particular function Ω and the particular point P_0 we have, by (42),

$$\begin{aligned}\frac{\left(\frac{\partial j}{\partial s}\right)_0}{j_0} &= \frac{[\Theta(f)]_0}{g_0^3} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0} \\ &= -\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0} - \frac{4\omega^2 \sin\phi_0 \cos\phi_0}{g_0}, \quad \text{by (45).}\end{aligned} \tag{50}$$

By Theorem 5 or relation (40) this expression (50) is the curvature at P_0 of the line of force, passing through P_0, of the positional field of force of potential function Ω. Hence by Theorems 4 and 6 we have

THEOREM 9. *The curvature at P_0 of the orthographic projection c' on the meridian plane of P_0 of the path of the falling particle is*

$$-\frac{1}{3}\frac{1}{g_0}\left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin\phi_0 \cos\phi_0\right]. \tag{51}$$

We have now found expressions for the curvature at P_0 of the curves d and c', which pass through the point P_0, lie in the meridian plane of P_0, and have (by Theorems 1 and 2) as common tangent at P_0 the vertical P_0T of P_0 (Fig. 2). By Theorem 7 we can now find expressions for the quantities ξ_1 and ξ_2 which were defined between equations (11) and (12). For this purpose let us denote by h the height of fall P_1P_0. In the first place, let us drop perpendiculars from P_1 and C' to the line P_0T and denote by $\overline{P}_1$ and $\overline{C}'$ respectively, the feet of these perpendiculars (Fig. 5). Then the infinitesimals $P_0\overline{P}_1$,

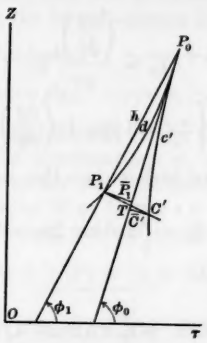

FIG. 5.

$P_0\overline{C}'$, $\overline{P}_1P_1$, $\overline{C}'C'$ differ from the infinitesimals $h = P_0P_1$, h, $\xi_2 = TP_1$, $\xi_1 = TC'$ respectively by infinitesimals of higher order. Therefore by Theorems 7, 8 and 9

$$\xi_2 = \pm\frac{1}{2}\left\{-2\frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}\right\}h^2 + \epsilon,$$

$$\xi_1 = \pm\frac{1}{2}\left\{-\frac{1}{3g_0}\left[\left(\frac{\partial g}{\partial \xi}\right)_0 + 4\omega^2 \sin\phi_0 \cos\phi_0\right]\right\}h^2 + \epsilon',$$

where ϵ and ϵ' are infinitesimals of order higher than that of h^2. In order to remove the ambiguity in sign let us observe that since the latitute ϕ lies between $-90°$ and $+90°$, it follows that $\cos\phi > 0$, and therefore, by relation (40), $(\partial j/\partial s)/j$ and d^2y/dx^2 are either both positive or both negative. Hence it follows from (45) and Theorem 8 that for the curve d, $-(\partial g/\partial \xi)_0/g_0$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative; and from (50) and Theorem 9 that for the curve c', $-1/g_0\{(\partial g/\partial \xi)_0 + 2\omega^2 \sin 2\phi_0\}$ and $(d^2z/d\tau^2)_0$ are either both positive or both negative. Furthermore, we assumed ξ_1 and ξ_2 to be positive toward the equator. Consequently if for definiteness we suppose (as shown in Fig. 5) that for curve d, $(d^2z/d\tau^2)_0 > 0$ and for curve c', $(d^2z/d\tau^2)_0 < 0$, it follows that $\xi_2 = -P_1T < 0$ and $\xi_1 = TC' > 0$. Therefore in the above expressions for ξ_2 and ξ_1 the lower signs must be used and thus we have

$$\xi_2 = \frac{\left(\frac{\partial g}{\partial \xi}\right)_0}{g_0}h^2, \tag{52}$$

$$\xi_1 = \left\{2\omega^2 \sin 2\phi_0 + \left(\frac{\partial g}{\partial \xi}\right)_0\right\}\frac{h^2}{6g_0}, \tag{53}$$

to terms of order not higher than the second in h, whence

$$\text{S.D.} = \xi_1 - \xi_2 = \left\{2\omega^2 \sin 2\phi_0 - 5\left(\frac{\partial g}{\partial \xi}\right)_0\right\}\frac{h^2}{6g_0}, \tag{54}$$

which is formula (I).

NOTE ON SOLVABLE QUINTICS.

BY PROFESSOR F. N. COLE.

(Read before the American Mathematical Society, January 2, 1915.)

THE substance of the following paper was included several years ago in my university lecture course on the theory of

groups. Being laid away for seasoning and final touches, it was destroyed in the fire of last October, along with the rather lengthy computations under Lagrange's theorem involved in the transition from the equations of Runge to those here employed. I have succeeded in restoring the paper from the lecture notes of my students, and am moved to publish it now in order that whatever value it may have may not be further exposed to destruction. The process of solution seems to be reduced to the simplest terms, and I have not been able to find that it has been anticipated.

The quintic is supposed to have been reduced (by Tschirnhausen transformation) to the form $x^5 + \alpha x + \beta = 0$. In his paper in the *Acta Mathematica*, volume 7 (1885), Runge obtained the sextic resolvent

$$(v - \alpha)^4(v^2 - 6\alpha v + 25\alpha^2) = 5^5\beta^4 v,$$

where $v = (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)^2$, the x's being the roots of the quintic, and expressed the coefficients of solvable (irreducible) quintics in terms of two parameters. The condition for solvability is of course that the resolvent sextic shall have a rational root. The function v belongs to the group of order 20, and the problem is substantially to calculate by radicals the four Lagrange functions

$$\tau_1 = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \epsilon^3 x_4 + \epsilon^4 x_5, \qquad \tau_3 = x_1 + \epsilon^3 x_2 + \epsilon x_3 + \epsilon^4 x_4 + \epsilon^2 x_5,$$

$$\tau_2 = x_1 + \epsilon^2 x_2 + \epsilon^4 x_3 + \epsilon x_4 + \epsilon^3 x_5, \qquad \tau_4 = x_1 + \epsilon^4 x_2 + \epsilon^3 x_3 + \epsilon^2 x_4 + \epsilon x_5.$$

These last belong to the identical group, their fifth powers to a group of order 5, the combinations $\sigma_1 = \tau_1^5 + \tau_4^5$, $\sigma_2 = \tau_2^5 + \tau_3^5$ to a group of order 10, and finally $5^3\rho = \sigma_1 + \sigma_2$ to the same group of order 20 as Runge's v.

I find now the following series of resolvent equations:

$$(1) \qquad 4^5\alpha^5(\rho^2 + 6\beta\rho + 25\beta^2) = (\rho + 25\beta)(\rho + 5\beta)^5;$$

$$(2) \qquad \sigma_1 + \sigma_2 = 5^3\rho, \quad \sigma_1\sigma_2 = -\frac{5^6}{2^4}(\rho^2 + 30\beta\rho + 125\beta^2);$$

$$(3) \qquad \begin{matrix} \tau_1^5 + \tau_4^5 = \sigma_1, \\ \tau_2^5 + \tau_3^5 = \sigma_2, \end{matrix} \qquad \tau_1^5\tau_4^5 = \frac{5^6}{2^6}\frac{(\rho + 25\beta)^3}{\sigma_2 - \sigma_1} = -\tau_2^5\tau_3^5.$$

As an example, the equation $x^5 + 15x + 12 = 0$ gives $\rho = 60$, $\sigma_1 + \sigma_2 = 5^4 \cdot 12$, $\sigma_1\sigma_2 = -5^8 \cdot 108$, $\sigma_1 = -5^4 \cdot 6$,

$\sigma_2 = 5^4 \cdot 18$. $\tau_1^5 + \tau_4^5 = -5^4 \cdot 6$, $\tau_1^5\tau_4^5 = 3^5 \cdot 5^5$, $\tau_1^5 = -1875 + 525\sqrt{10}$, $\tau_4^5 = -1875 - 525\sqrt{10}$. $\tau_2^5 + \tau_3^5 = 5^4 \cdot 18$, $\tau_2^5\tau_3^5 = -3^5 \cdot 5^5$, $\tau_2^5 = 5625 + 1800\sqrt{10}$, $\tau_3^5 = 5625 - 1800\sqrt{10}$. $5x = \tau_1 + \tau_2 + \tau_3 + \tau_4$.

The connection with Runge's resolvent is effected by the relation

$$\rho = -5\beta\frac{v - 5\alpha}{v - \alpha},$$

by which equation (1) may be verified. The relation

$$\sigma = \frac{-5^4\beta}{2(v - \alpha)}(v - 5\alpha + \sqrt{v^2 - 6\alpha v + 25\alpha^2}),$$

which includes the preceding and gives the key to equations (2) and (3), was worked out by Lagrange's theorem.

COLUMBIA UNIVERSITY,
April 6, 1915.

THE MADISON COLLOQUIUM LECTURES ON MATHEMATICS.

Part I: On Invariants and the Theory of Numbers. By LEONARD EUGENE DICKSON. New York, American Mathematical Society, 1914.

THE number of new mathematical systems which may be characterized as distinct mutations, whose discovery or development is to be credited to American research, has shown a marked increase within a few decades. The reviewer of Professor Dickson's Lectures of the Madison Colloquium volume has the satisfaction of recording one of these great discoveries, his theory of classes in invariant theory, and of observing how as a result of this discovery, number theory, which long had little contact with the theory of invariants, now has very much in common with it. Dickson's technical memoirs in which the theory of classes and the invariant theory of modular forms were first expounded appeared in 1909. And while the material and indeed much of the method also of the Colloquium Lectures are new, they are dominated by the theory of classes and may, therefore, be regarded as a superstructure of the system founded in his 1909 papers. Lecture I may be regarded, also, as introductory to the theory as a whole.

The theory of classes is a general invariant theory, applicable, and applied by the lecturer, to three types of invariants. We now define these. Ordinary invariant theory of algebraical quantics, which we may call the *algebraic* theory, is one in which the coefficients of the linear transformations as well as the coefficients of the transformed quantics themselves are perfectly arbitrary variables. If both of these sets of coefficients are parameters representing residues of a prime number p (or, generally, marks of a Galois field), we have the invariant theory called *modular* invariant theory, due to Dickson. Again if the coefficients of the quantics are variables while the coefficients of the transformations are modular the invariant formations are called *formal modular*. This type of invariant was first defined by A. Hurwitz.

Suppose that S is a system of modular forms, and to follow the inductive plan of the Lectures, suppose that the modulus is p and that S consists of one modular quadratic form q_m in m variables. The coefficients β_{ij} of q_m are parameters to which may be assigned in turn particular sets of residues modulo p, giving the particular forms q_m', q_m'', $\cdots$. Now if the totality of forms $q_m^{(k)}$ be transformed by all of the transformations of the linear group L (mod p) on the variables, the $q_m^{(k)}$ are separated into classes C_i such that two particular forms belong to the same class if and only if they are equivalent under L.

The definition of an invariant now becomes a function-theoretic matter. A single-valued function φ of the undetermined coefficients β_{ij} is an invariant of q_m if φ has the same value for all sets β_{ij}', β_{ij}'', $\cdots$ of coefficients of forms q_m', q_m'', $\cdots$ belonging to the same class.

To determine the value of an invariant φ for a given class C_i we need only assign to the β_{ij} in φ the particular coefficients in a canonical form of q_m which belongs to C_i. In consequence invariants may here be determined from their values by an interpolation formula, or by some particular method whose use would be equivalent to the determination of a function by the interpolation process. And to determine a fundamental system of invariants we need only to determine a set which completely characterizes the classes C_i. Specifically a set of invariants $\varphi_1, \varphi_2, \cdots$ is said to characterize completely the classes when each φ_k has the same value for two classes only when the latter are identical, and the following theorem

proved in Lecture I, taken with the interpolation idea, furnishes not only a powerful construction method, but also an advantageous means of proving that a system which has been constructed forms a fundamental system: If the modular invariants $A, B, \cdots, J$ completely characterize the classes, they form a fundamental system of modular invariants.

The particular problems treated in Lecture I are, first, the reduction of the algebraic quadratic form

$$q_m = \sum_{i,j=1}^{m} \beta_{ij} x_i x_j \qquad (\beta_{ij} = \beta_{ji})$$

to the canonical form

$$x_1^2 + \cdots + x_{m-1}^2 + D x_m^2 \qquad (D = |\beta_{ij}|), \tag{1}$$

or to

$$x_1^2 + \cdots + x_r^2, \tag{2}$$

according as the rank r of D is $= m$ or $< m$. This can be done by linear transformations with complex coefficients of determinant unity, and thus all algebraic quadratic forms may be separated into the classes

$$C_{m,D},\ C_r \quad (D \neq 0,\ r = 0, 1, \cdots, m-1),$$

where, for a particular value of D, $C_{m,D}$ is composed of all forms q_m of determinant D, each being transformable into (1), and so on. Every single-valued algebraic invariant of q_m is a single-valued function of D and r which completely characterize the classes.

Secondly, the corresponding canonical reduction of the modular q_m is made; the forms are (1), (2) and

$$x_1^2 + \cdots + x_{r-1}^2 + \rho x_r^2 \qquad (0 < r < m), \tag{3}$$

and the classes

$$C_{m,D},\ C_{r,+1},\ C_{r,-1},\ C_0 \quad (D=1, \cdots, p-1;\ r=1, \cdots, m-1),$$

where, for instance, $C_{r,-1}$ is composed of all forms transformable into (3). A fundamental system of rational integral modular invariants of q_m is then

$$D, A_1, \cdots, A_{m-1}, I_0,$$

where if $M_1, M_2, \cdots, M_n$ denote the principal minors of order r of D, and d ranges over the principal minors of orders $> r$,

$$A_r = \{M_1^{(p-1)/2} + M_2^{(p-1)/2}(1 - M_1^{p-1}) + \cdots$$

$$+ M_n^{(p-1)/2}(1 - M_1^{p-1}) \cdots (1 - M_{n-1}^{p-1})\} \times \Pi(1 - d^{p-1}).$$

This invariant has the value + 1 for any form of class $C_{r,1}$, for example. Also $I_0 = \Pi(1 - \beta_{ij}^{p-1})$.

In the opening paragraphs of Lecture II the algebraic binary quartic forms f are separated into classes by transformations of the type (algebraic) $T: x = x' + ty', y = y'$. Since invariants under T are seminvariants of f we arrive at a determination of five seminvariants of f from the point of view of the classes, and a proof that they form a fundamental system of rational integral seminvariants. This set does not completely characterize the classes, i. e., is not a fundamental system of *single-valued* seminvariants of f. If f is modular and $p > 3$ eight determinate seminvariants characterize the classes.

Professor Dickson next establishes an inductive method of constructing all of the members of a fundamental system of modular seminvariants of a form of order n from the system for a form of order $n - 1$. For instance if n is divisible by p $(n = pq)$, and $F_n = A_0x^n + A_1x^{n-1}y + \cdots$, the set consisting of A_0 and the fundamental system for

$$\bar{F}_{n-1} = \frac{1}{y}(F_n - \varphi) \quad [\varphi = A_0(x^p - xy^{p-1})^q] \tag{4}$$

completely characterize the classes of F_n.

By this and similar processes an explicit fundamental system of modular seminvariants of F_n $(p > n)$ is constructed and for particular low orders some explicit systems for $p \gtrless n$. Invariants are then treated as seminvariants which possess the right type of symmetry, and the subject of linearly independent sets claims attention.

Lecture III is devoted to concomitants of the formal modular type. The first problem solved is that of the determination of a set of rational integral invariant functions of the variables of the modular transformations L, alone, such that any other such function is a rational integral function of those determined, with integral coefficients; i. e., the determination of a fundamental system of universal covariants of the group L. The first such determination was made by Dickson in a paper published in 1911. But for the case of two variables the work is here simplified by the introduction of geometrical concepts.

The transformations being

$$(5)\quad G\colon x' \equiv bx + dy,\quad y' \equiv cx + ey,\quad be - cd \equiv 1 \pmod{p},$$

and a point being defined as a pair of homogeneous coordinates $(x, y) = (kx, ky)$, we say that a point is a special point if it is invariant under at least one transformation which is not identity. For it

$$x' \equiv \rho x,\quad y' \equiv \rho y,$$

and ρ is a root of the characteristic congruence

$$(6)\qquad \rho^2 - (b + e)\rho + 1 \equiv 0 \pmod{p}.$$

Only real special points are invariant when (6) has an integral root, and all real points are conjugate under G. It follows that if an invariant of G vanishes for one of the real points it vanishes for all and has the factor

$$L = y\prod_{a=0}^{p-1}(x - ay) \equiv x^p y - xy^p \pmod{p}.$$

If (6) has Galois imaginary roots, the corresponding invariant, representing the conjugate set of imaginary special points, is

$$Q = (x^{p^2}y - xy^{p^2}) \div L,$$

and L, Q is the system sought.

The details of the proof here contain the two principal elements in the author's main method of constructing formal modular invariants and showing what ones are reducible. For with $p = 2$ and $f = ax^2 + bxy + cy^2$ the transformation $x = x' + y'$, $y = y'$ induces the transformation

$$a' \equiv a,\quad b' \equiv b,\quad c' \equiv a + b + c \pmod{2}.$$

This latter may be identified with a special case of (5), and the two universal covariants of this special group become formal modular seminvariants of f. As to reducibility, in view of the theory of conjugate points, we need only show that a covariant has the factor y in order to know that it has the factor L.

Fundamental systems of formal modular seminvariants and invariants of the binary quadratic form modulo p are completely determined in this Lecture, as well as sets for the cubic, and some simpler modular covariant systems modulo 2.

Brief mention is given to the total binary group's form problem, and the invariantive classification of forms.

We now come to the two lectures on modular geometry, and in Lecture IV, the modular geometry and covariantive theory of a quadratic form in m variables modulo 2. Modular geometry is not a new term, but Dickson's formulation of a new theory under the old name marks a notable advance. In modular theories such as G. Arnoux's Essai de Géométrie analytique modulaire (1911), a "curve" is a finite aggregate of real points. Thus an analytic representative of this curve, its quantic, is not determinate in any sense from its points. Dickson assumes at the beginning that the curve, represented for instance by the modular form

$$q_m(x) = \Sigma c_{ij}x_ix_j + \Sigma b_ix_i^2 \quad (i, j = 1, \cdots, m; i < j),$$

modulo 2, shall contain an infinitude of points. That is, he defines a point as a set of m ordered elements $(x_1, \cdots, x_m)$, not all zero, of the infinite field F_2 composed of the roots of all congruences modulo 2 with integral coefficients. The point $(x) = (x_1, \cdots, x_m)$ is called real if the ratios of the x's are congruent to integers modulo 2, otherwise it is imaginary. Then the aggregate of points (x) for which $q_m(x) \equiv 0 \pmod 2$ is called a quadric locus, a conic if $m = 3$. The quadric is thus composed of an infinitude of points, only a finite number of which are real. The investigation of the modular invariant theory of this locus is carried out as a purely arithmetical theory, without any geometrical representation of the locus, although the terminology and to some extent the methods of analytic projective geometry are employed. The lack of any mode of geometrical representation leaves the reader with a feeling of conjecture as to just what kind of geometry in the concrete he is here concerned with, and as to whether something similar, perhaps, to isometric projection could be invented to give a picture of the infinite point cluster constituting the modular curve.

The first invariant formation treated in this lecture is the polar locus

$$P(y, z) \equiv \Sigma c_{ij}(y_iz_j + y_jz_i) \pmod 2.$$

For an odd m the polars of all points (y) have at least one point in common. A determinate common point, whose coordinates happen to be cogredient to the variables, is called the apex of the quadric. Any line through the apex is tangent to the

quadric, and conversely. Thus the quadric has a linear tangential equation. If (C) is the apex, $q_m(C)$ is a formal modular invariant. If it vanishes, the apex is on the locus, which is then a cone. In the case of a conic, reducible to

$$\varphi \equiv X_1X_2 + X_3^2,$$

the only real points on the locus are (1, 1, 1), (1, 0, 0), (0, 1, 0). The apex is (0, 0, 1). The only other real points, three in number, lie on the covariant line

$$X_1 + X_2 + X_3 \equiv 0 \pmod 2. \tag{7}$$

A like configuration on the real points defined by the quinary surface, of great beauty, is constructed, and a similar theory for the case of an even m is given.

The latter half of this chapter, the most technical part of the Lectures, is devoted to a determination, from the standpoint of the classes, of a fundamental system of modular covariants of the ternary quadratic form F with integral coefficients modulo 2.

With a modular analytic projective geometry defined and its covariant theory established Professor Dickson proceeds in Lecture V to a particular curve and a particular feature of the geometry on that curve. This is a theory of plane cubic curves with a real inflexion point which holds true both in ordinary and in modular geometry. After reducing the cubic to the normal form

$$C = x^2y + gy^3 + hy^2z + \delta z^3 \qquad (\delta \neq 0),$$

the author develops the theory of the inflexions. A prominent part is played by the invariants

$$s = -3\delta h, \quad t = -108\delta^2 g.$$

A cubic with integral coefficients taken modulo p, a prime > 3, with at least one real inflexion point and with invariant $s = 0$, and $t \neq 0$, has nine real inflexion points if p is of the form $3j + 1$ and $-t$ is a sixth power modulo p, a single real inflexion point if $p = 3j + 1$ and $-t$ is a quadratic nonresidue of p, and exactly three real inflexion points in all other cases. The author proves a series of similar theorems.

As a whole these Lectures are indeed a most meritorious contribution, suggesting many new problems of many new kinds.

O. E. GLENN.

SOME BOOKS ON CALCULUS.

Elements of the Differential and Integral Calculus (Revised Edition). By W. A. GRANVILLE. Edited by P. F. SMITH. Boston, Ginn and Company, 1911. xv+463 pp.

Elementary Textbook on the Calculus. By V. SNYDER and J. I. HUTCHINSON. New York, American Book Company, 1912. 384 pp.

The Calculus. By E. W. DAVIS assisted by W. C. BRENKE. Edited by E. R. HEDRICK. New York, The Macmillan Company, 1913. xx+383+63 pp.

Esercizi di Analisi Infinitesimale. By G. VIVANTI. Pavia, Mattei, 1913. vii+470 pp.

GRANVILLE'S Calculus is too widely known both in its original and in its revised edition to require any long notice.* A number of changes have been introduced in the revision and all seem to improve the work as a class drill book. In the number of pages the additions and subtractions exactly balance.

In the preface the author states that in the last few years considerable progress had been made in the teaching of the elements of the calculus and in this revised edition the latest and best methods are exhibited. This statement is entirely incomprehensible to us. So far as we have observed the only important improvement in teaching calculus has been to introduce the calculus earlier in the student's course and so to present it in matter as in time that it may be of greater use to the student in his courses on physics and mechanics. Granville's book veers not the slightest toward this point, no more in the revised than in the original edition.

* For a review of the original see E. B. Van Vleck, this BULLETIN, volume 12, pages 181–187. We are personally out of sympathy with that review because we believe that it represents the view-point of the mathematician catering to the one per cent of the students of calculus who will possibly be pure mathematicians rather than the point of view of the teacher of mathematics who sets his heart on doing the maximum good to the maximum number and who regards mathematics through calculus as essentially the handmaiden of the theoretical and applied sciences. We believe that the aristocratic movement has passed its zenith and is giving way to a less selfish and more democratic point of view,—and we daresay the earlier reviewer is in sympathy with the change.

The calculus is taught to such a large number of students in so many institutions that there is no particular reason why any teacher who has a prominent position cannot find or should not find a publisher for his own notes on calculus and thus have a text of his own which suits him better than any other. This is sufficient excuse for the appearance of Snyder and Hutchinson's book. The work is short. It could have been made shorter without harm by abridging the 42 pages given to contact and curvature, singular points, and envelopes. The most natural book with which to compare Snyder and Hutchinson's is Osborne's (revised edition, 1908). The two are a good deal alike; they give the calculus which is really needed and give it in direct teachable form,—which must be balm to the souls of those that are bored by the modernization of calculus toward rigor, or toward "practical mathematics," or toward the so-called "mixed method."

In their preface the authors call attention to the pressure toward shortening the course in mathematics, they cite the appearance of books on calculus for engineers, physicists, chemists, and so on, and state that it is in recognition of this pressure that they have written. It is good that they are alive to the advisability of adapting calculus to the students who take it; we should all be alive to that fundamental principle of educational justice. But is there any real pressure to shorten the course in mathematics? Is not the pressure rather to get the kind of mathematics the student, in the opinion of engineers, etc., needs? And there is plenty of that kind. Is not the shortening merely an indirect result due to the fact that we will not give the student that which others think he needs and that they therefore diminish his time with us so that they may give him what, in their opinion, he needs more than what we would offer him in any additional time allowed to us?

We may quote from the introduction to Perry's Elementary Practical Mathematics: "Academic methods of teaching mathematics succeed with about five per cent of all students, the small minority who are fond of abstract reasoning; they fail altogether with the average student. Mathematical study may be made of great value to the average man if only it is made interesting to him." Here is the real reason for the pressure there is upon us. We deal in the abstract and in the rigorous; the average person does not, and to a certain

extent cannot. We teach the wrong way,—let us quote again: "There is always a difficulty in obtaining competent teachers (of practical mathematics). Any man who has learnt pure mathematics is thought by himself and others to be fit to teach, whereas his very fondness for and his fitness to study pure mathematics make it difficult for him to understand the simple principles underlying the new method. The average boy cannot take to abstract reasoning, and he is called stupid; I think him much wiser than the boy who is usually called clever."

We may not believe any of this stuff, we may force it out of our consideration; but there are many who believe it all, and they will constantly bring it back to our attention. And we cannot compromise more than temporarily by abridging our course; the very abridgment will produce less efficiency in the sort of thing we do teach. Even an average class will take great delight in hard differentiations and integrations, they will rejoice in conquering the difficulty, as I many times observed in the classes of A. W. Phillips at Yale,—provided the class is drilled in differentiation and integration until the majority acquire sufficient technique to make the game interesting. It is ability to do that maintains the interest. When we abridge our course without otherwise changing it we diminish the chances that the student shall become able to do what we teach him. That is the weakness of mere abridging. Diminishing the requirement in Greek for entrance to college killed preparatory Greek as quickly as anything could.

For ourselves, we do not believe in going the whole way with Perry; we believe that some abstract reasoning is good, and with our students prepared as they are when they come to us from the secondary schools a certain amount of abstract reasoning is not only good but possible. If we can follow a short course in calculus from a book like Snyder and Hutchinson's by a considerable course in concrete and practical problems, that may be our best procedure. But if we are to be allowed altogether only a short course, we should make that much less mathematical in the canonical sense; and by doing so we may perhaps be entrusted with a greater allotment of time.

Davis's Calculus is a frank attempt to introduce variety

and interest into the calculus. The work therefore has attractive elements; one may easily exclaim: How inspiring to teacher and pupil to have all this constant contact with nature! That the book has bad qualities is obvious to anybody examining it carefully, but it is only after the sad disillusionment of teaching it a year that one can really find out how largely the bad outweighs the good. The book will therefore have many enthusiastic adopters and many speedy rejectors.

The main difficulty is that careless workmanship (or playmanship) permeates the whole in such an insidious fashion that it is partly hidden to the prospective user and always a burden to the actual user. Whether author, assistant author, or editor is responsible for this defect we cannot say; but it is improbable that any real hard cooperation by all upon the whole could have left so many lesions, and we may guess that one brewed the text, another peppered in the exercises, and a third sprinkled in the sage advice to Dear Reader and the gratuitous reflections. The answer book is full of errors, and thus is a great annoyance to the serious student, a corrupter of the careless worker. A table is valueless except as it is accurate, yet inaccuracies are found in the formula for center of pressure (not given in the text) and in the polar equation of the cissoid.

If Davis-Brenke-Hedrick had written a sufficiently original text we could pardon a number of errors, even under triple responsibility; but there is no particular originality about the work. They treat the algebraic function first, both as regards integration and differentiation, and when they come to transcendental functions they carry on the differential and integral calculus simultaneously. But so did Mercer in 1910; and if we may trust a comparative judgment of two books one of which we have not taught, we should unhesitatingly say that Mercer, though bearing but one workman's name, is incomparably the more careful and valuable production. And, to mention no others, Byerly in his Differential Calculus as long ago as 1879 introduced the integral calculus early and carried it along with the differential. It can hardly be expected that Byerly's book as it stands after 35 years should appeal strongly to teachers of the present day; yet its plan has many of the good features of recent books which try to freshen up the calculus.

The authors include a considerable treatment of differential

equations as did Mercer in 1910 and Woods and Bailey in their Course in Mathematics, volume 2, some years before. This is not a bad thing to do, for the integration of the simpler differential equations is a natural part of the integral calculus; but I am very strongly of the opinion that a systematic chapter on rectilinear dynamics, such as is found in Byerly's and Osgood's books, is of much more value to the student and teaches him just as much about differential equations. One of the earliest differential equations we meet is the so-called compound interest law, for which our authors use the term snowball law as more "suggestive"; it is too bad that they regard it as more suggestive, for it is very wide of the mark. (The snowball law makes an interesting exercise for engineers; it leads under reasonable assumptions, such as an engineer would make, to an algebraic integral.)

One thing above all others students of mathematics should learn, namely, that it is the business of mathematics to teach them to think and talk coherently. They should have clean-cut definitions and straightforward proofs. If a book contains these, the teacher may safely be trusted to furnish the discussion and to point out the beauties of the landscape; whereas it is difficult for the teacher to furnish the definition and proof with any effectiveness to a class whose text is mostly discussion. The authors' book is long despite its moderate number of pages; for it is printed to a large extent in eight-point type. It could be abridged without substantive omissions.

It is the business of sets of exercises to offer definite and somewhat graded problems to the student; but the exercises in this text are wholly ungraded and in many cases very indefinite in statement. Sometimes there is scarcely a single exercise in a whole set which I should wish to assign to a class, and in every case it is necessary to take the greatest care in the selection of exercises. We recommend a careful reading of the lists of exercises to every prospective user.

The practice of leaving important items for exercises, a plan of which I heartily approve in advanced texts where exercises are hard to find or where we are trying to teach the student research, seems very doubtful for any elementary text. Yet here the area of a surface of revolution, component accelerations, hyperbolic functions, average values, centers of gravity, the element of volume in polar or spherical coordinates are thus left as exercises. It must be a remarkable class that

can work out for itself the polar element of volume; it is more than most of us can do to draw a decent figure and give the proof for the class.

We have heard a great deal off and on about the necessity of giving the student power and the spirit of investigation; but this is merely a visionary's ideal, as anybody can see by pondering upon the question: How many of our doctors of philosophy in mathematics in this country or abroad are engaged in real research? If six to eight years of training lapse into desuetude in the case of professional mathematicians, what can you expect to accomplish with sophomores? Put into the text what you want them to know in such form that they can learn it, say I, and then see that they do learn it. And I have heard a very eminent investigator recommend the same sort of thing for candidates for the doctorate.

There is no need of going into the details, whether bad or good, of Davis's text. Suffice it to say that if books are not more carefully written, we shall have to refrain from adopting them from very lack of time to examine them in sufficient detail to make it safe to adopt them; but it is too bad to throw the whole reponsibility upon the user instead of upon the author and publisher, where we previously thought it belonged, at least to a very large extent.

Vivanti's book of exercises, a companion to his Lezioni d'analisi infinitesimale, contains 575 well-selected solved exercises in calculus; there are no applications and no rigorous types. The list should be of value to American writers of texts, but it is difficult to see how it can be of direct use in our classes.

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NOTES.

THE April number (volume 16, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Quartic curves modulo 2," by L. E. DICKSON; "Mixed linear integral equations of the first order," by W. A. HURWITZ; "Prime power groups in which every commutator of prime order is invariant," by W. B. FITE; "On

the order of primitive groups," by W. A. MANNING; "A proof of the invariance of certain constants of analysis situs," by J. W. ALEXANDER, II; "Point sets and allied Cremona groups," by A. B. COBLE; "Scroll directrix curves," by C. T. SULLIVAN.

THE April number (volume 37, number 2) of the *American Journal of Mathematics* contains: "Invariantive theory of plane cubic curves modulo 2," by L. E. DICKSON; "On the projective differential classification of n-dimensional spreads generated by ∞^1 flats," by ARTHUR RANUM; "On the order of a restricted system of equations," by F. F. DECKER; "One-parameter families of curves," by L. P. EISENHART; "On a porism connected with the theory of Maxwell's equations and a method of obtaining the lines of electric force due to a moving point charge," by H. BATEMAN; "The abstract definitions of groups of degree 8," by JOSEPHINE E. BURNS.

THE Association of mathematics teachers of New Jersey held its second regular meeting at Trenton on May 1. Papers were read by OSWALD VEBLEN: "The affine geometry"; RICHARD MORRIS: "The auxiliary angle"; FLETCHER DURELL: "Mathematics and efficiency"; A. W. BELCHER: "First year mathematics for a technical high school"; C. O. GUNTHER: "Trigonometry for the college student"; J. W. COLLITON: "The study conference plan in mathematics"; H. E. WEBB: "Geometric definition of the trigonometric functions" and "Outline of a course in advanced commercial algebra." Professor H. B. FINE was elected president of the association.

THE Adams prize for the year 1913–1914 has been awarded to G. I. TAYLOR (Trinity, 1910). The title of the prize memoir is: "The phenomena of the disturbed motion of fluids, including the resistances encountered by bodies moving through them." The value of the prize is about £250.

THE early publication of an Analytic Mechanics, by J. A. MILLER and S. B. LIBBY, of Swarthmore College, is announced by D. C. Heath and Company.

UNIVERSITY OF CALIFORNIA.—The following advanced courses are announced for the summer session, June 21 to July 31:—By Professor M. W. HASKELL: Constructions by

ruler and compass.—By Professor C. J. KEYSER: Analytical geometry of three dimensions; History and significance of central mathematical ideas and doctrines.—By Professor H. W. TYLER: History of mathematical science.—By Professor T. M. PUTNAM: Three famous problems of geometry; Theory of functions of a complex variable.

DR. E. S. ALLEN, of Brown University, has been appointed instructor in mathematics at the University of Michigan.

PROFESSOR J. J. HARDY, who had held the chair of mathematics and astronomy at Lafayette College since 1876, died May 2, 1915, at the age of seventy-one years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAJORI (F.). History of mathematics. 2d reprint. New York, Macmillan, 1913. 8vo. 422 pp. $3.50

CAMPBELL (D. F.). A short course on differential equations. 5th reprint. New York, Macmillan, 1913. 12mo. 123 pp. $0.90

——. Elements of the differential and integral calculus. 4th reprint. New York, Macmillan, 1913. 12mo. 362 pp. $1.90.

CARVALLO (E.). Le calcul des probabilités et ses applications. Paris, Gauthier-Villars, 1912. 8vo. 9+169 pp.

CREMONA (L.). Opere matematiche. Tomo II. Milano, Hoepli, 1915. 4to. 459 pp. L. 25.00

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band II 2, Heft 4: E. Hill, Lineare Differentialgleichungen im komplexen Gebiet. Leipzig, Teubner, 1915. Gr. 8vo. Pp. 471–562. M. 2.80

FINE (H. B.) and THOMPSON (H. D.). Coördinate geometry. 5th reprint. New York, Macmillan, 1914. 12mo. 300 pp. Half leather. $1.60

HAEUSSLER (J. W.). Geometrischer und algebraischer Beweis des Fermatschen Satzes durch Ausziehen der n-ten Wurzel und durch Ableitung des Satzes als ganzzahlige Ungleichung. Berlin, M. Krayn, 1912. 8vo. 48 pp.

HILL (E.). See ENCYKLOPÄDIE.

LA MARCA (R.). Criteri di congruenza e criteri di divisibilità. Esercizi vari. Torre del Greco, Pantaleo, 1912. 8vo. 30 pp.

LASKA (W.). Einleitung in die geometrische Funktionentheorie. 2te Auflage. Bremerhaven, 1915. Gr. 8vo. 6+87 pp. M. 2.00

NEKRASSOW (P. A.). Theory of probabilities. 2d edition, with the addition of the statistical theory of correlation and the elements of nomography. (In Russian.) St. Petersburg, 1912. 8vo. 36+532 pp.

RIGGS (N. C.). Analytic geometry. 8th reprint. New York, Macmillan, 1914. 12mo. $1.60

THOMPSON (H. D.). See FINE (H. B.).

WIESNER (S.). Ueber mathematische Logik. Berlin, 1914. Gr. 8vo. 12 pp.

II. ELEMENTARY MATHEMATICS.

BEARD (W. S.). Arithmetical examples. London, Dent, 1912. 269 pp.

COMFORT (L. C.). Geometry: a reasoned chain. London, Gill, 1915. 1s.

CONTI (A.). Elementi di calcolo letterale. 3a edizione. Bologna, Zanichelli, 1912. 16mo. 136 pp.

HALL (H. S.) and STEVENS (F. H.). Examples in arithmetic taken from "A school arithmetic." London, Macmillan, 1912. 8vo. 15+281+39 pp.

HOČEVAR (F.). Lehr- und Uebungsbuch der Arithmetik für Gymnasien und Realgymnasien. Oberstufe. 2te umgearbeitete Auflage. Wien, F. Tempsky, 1912. 152 pp.

JENSEN (O.). See WALTHER (F.).

LESSER (O.). See SCHWAB (R.).

NASAREVSKY (J.). Die Anzahl der Ziffern in der Periode des periodischen Dezimalbruches (Vortrag gehalten in der Sitzung der Mathematischen Gesellschaft Charkow 15/28. IV. 1909). Charkow, 1912. 8vo. 6 pp.

OTTO (F.), PETRI (W.), THAER (A.) und ZIEGLER (J.). Mathematik für Oberlyzeen. In 2 Teilen. 1ter Teil unter Mitwirkung von Otto, bearbeitet von Petri, Thaer, Ziegler. 2te umgearbeitete Auflage. Leipzig, F. Hirt, 1912. 8vo. 7+293 pp.

PETRI (W.). See OTTO (F.).

ROTHROCK (D. A.). Elements of plane and spherical trigonometry, with tables. 3d reprint. New York, Macmillan, 1912. 8vo. 147+99 pp. $1.40

ROUWOLF (R.) See THAER (A.).

SCHWAB (R.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an höheren Lehranstalten. 2te Auflage. Leipzig, Freytag, 1912. 8vo. 290 pp.

SOPOTSKO (L.). The accuracy of computation with logarithmic tables. (In Russian.) Moscow, 1912. 8vo. 12pp.

SPORER (B.). Niedere Analysis. 2te Auflage. 4ter Abdruck. Berlin und Leipzig, Göschen, 1912. 8vo. 179 pp.

STEVENS (F. H.). See HALL (H. S.).

THAER (A.). See OTTO (F.).

THAER (A.) und ROUWOLF (R.). Rechenbuch für höhere Schulen. Ausgabe B, in drei Heften und einem Ergänzungsheft. 1tes Heft, für Sexta, 3te und 4te Auflage, 1913; 2tes Heft, für Quinta, 3te und 4te Auflage, 1914; 3tes Heft, für Quarta und Untertertia, 3te und 4te Auflage, 1914; Ergänzungsheft, für Obertertia und Untersekunda, 2te Auflage, 1913. Breslau, F. Hirt. 95+96+96+100 pp. Kartoniert. M. 1.00+1.00+1.00+1.00

——. Rechenbuch für Vorschulen höherer Lehranstalten in drei Heften. 3te Auflage. Breslau, F. Hirt, 1914. 83+95+100 pp. Kartoniert. M. 0.80+0.90+1.00. Ergebnisse zum 3ten Heft. M. 0.60

THAER (A.) und WIMMENAUER (T.). Arithmetische Aufgaben für höhere Schulen. Ausgabe C, für Realschulen. Breslau, Hirt, 1912. 8+160 pp. Geb. M. 2.40. Ergebnisheft dazu. Geh. M. 1.60.

TOROPOV (K.). Magic series and their application to the solution of exercises. (In Russian.) 2d edition. Orenburg, 1912. 8vo. 45+2 pp.

WALTHER (F.) und JENSEN (O.). Mathematischer Lehr- und Uebungsgang für höhere Mädchenschulen, Lyzeen und Studienanstalten. Abtl. A 3 (Kl. VII–V). 3tes Heft (Kl. V) von Jensen. Leipzig, F. Brandstetter, 1912. 8vo. 140 pp.

WIMMENAUER (T.). See THAER (A.).

ZIEGLER (J.). See OTTO (F.).

III. APPLIED MATHEMATICS.

BIÉMONT (P.). See CÉSAR (J.).

CALDARERA (F.). Corso di meccanica razionale. Vol. I: preliminari, cinematica, studio delle forze. 2a edizione migliorata. Palermo, Virzi, 1915. 8vo. 8+336 pp. L. 13.00

CASTLE (F.). Workshop arithmetic. London, Macmillan, 1915. 1s. 6d.

CÉSAR (J.) et BIÉMONT (P.). Traité d'arithmétique financière. Paris, Hermann, 1912. 8vo. 170+36 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band IV 2 II, Heft 5: K. Wieghardt, Theorie der Baukonstruktionen, II; E. Hellinger, Allgemeine Ansätze der Mechanik der Kontinua; T. Karman und L. Föppl, Physikalische Grundlagen der Festigkeitslehre. Leipzig, Teubner, 1914. Gr. 8vo. Pp. 15+535–770. M. 8.00

FESTSCHRIFT W. Nernst zu seinem 25 jährigen Doktorjubiläum gewidmet von seinen Schülern. Halle, W. Knapp, 1912. Gr. 8vo. 6+487 pp.

FOCK (E.). See KLESSNER (E.).

FÖPPL (L.). See ENCYKLOPÄDIE.

HELLINGER (E.). See ENCYKLOPÄDIE.

IWERONOV (I.). Die Methode der kleinsten Quadrate. Vorlesungen gehalten im Konstantinischen Feldmessinstitut und Moskauer Agronomischen Institut. 2te Auflage. Moscow, 1912. 8vo. 136 pp.

KARMAN (T.). See ENCYKLOPÄDIE.

KLESSNER (E.) und FOCK (E.). Physik und Chemie für Lyzeen und höhere Mädchenschulen sowie für den Anfangs-Unterricht in Studienanstalten. Heft 1: Mechanische Grundbegriffe; Mechanik starrer Körper, 1ter Teil; Wärmelehre. Berlin, Salle, 1914. 71 pp. Kartoniert. M. 1.50

MANSION (P.). L'avantage du banquier au jeu de baccara. Louvain, Ceuterick, 1912. 12mo. 8 pp.

NERNST (W.). See FESTSCHRIFT.

THIRY (C.). Etude pratique et usuelle des emprunts, obligations. Gand, Van Goethem, 1912. 8vo. 76 pp.

VAUCHER (A.). Notes additionelles à la théorie de l'échelle musicale. Paris, Gauthier-Villars, 1914. 8vo. 16 pp. Fr. 0.75

WIEGHARDT (K.). See ENCYKLOPÄDIE.

ZAUBRECHER (F. X.). Die Lebensversicherung. (Sammlung Kösel, Band 53.) Kempten, J. Kösel, 1912. Sm. 8vo. 12+202 pp.